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Fixed Point Theorems for compatible mappings of Type (A) for four mappings satisfying contractive conditions of integral type in metric spaces

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ABSTRACT

In this paper, we prove fixed point theorems for compatible mappings of type (A) for four mappings satisfying contractive conditions of integral type in metric spaces. In this paper, we extend and generalized the results of Aage and Salunke [1], Branciari[2], Murthy[5], Pathak and Khan[6]and Sharma and Sahu [10] for generalized contraction of integral type.

Keywords: fixed point, compatible mappings, compatible mappings of type (A).

1. INTRODUCTION AND PRELIMINARIES

The concept of weakly commuting mappings was introduced by Sessa[8] and some fixed point theorems was obtained in complete metric space . Compatible mappings was defined by Jungck[3] and few fixed theorems was discussed in complete metric space .Also he proved that weak commuting mappings are compatible mappings but converse need not be true. Further the new concept i.e. compatible mapping of type (A) was introduced by Jungck et al.[4] and some fixed point theorems was proved in metric space. Compatible mappings of type (A) are more general than weakly commuting mappings and converse need not be true. The concept of A-compatible and S-compatible by splitting the definition of compatible mapping of type (A) was introduced by Pathak and Khan [6].

Some fixed point theorem for compatible mappings of type (P) was proved by Pathak et al.[7], as application they prove the existence and uniqueness problem of common solution for a class of functional equations arising in dynamic programming. Recently, fixed point theorems of A- compatible and S-compatible mappings were proved by Shahidur Rahman et al. [9] and generalized the result of Murthy [5], Sharma and Sahu [10]. Some fixed point theorem for compatible mappings of type (A) for four self mappings of a complete metric space was proved by Aage and Salunke [1]. Recently, Banach contraction principle for integral type contraction was proved by Branciari [2].

There are following definitions of different types of compatible mappings.

Definition 1.1[8]: Self-mappings S and T of a metric space (X, d) are said to be **weakly commuting** pair iff $d(STx, TSx) \leq d(Sx, Tx)$ for all $x \in X$. Clearly, commuting mappings are weakly commuting but converse is not true.

Definition 1.2[3]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be **compatible** if $\lim_{n\to\infty} d(ASx_n,SAx_n)=0$, Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Ax_n= \lim_{n\to\infty}Sx_n = t$ for some $t \in X$.

Definition 1.3[4]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be **compatible of type (A)** if

$$\lim_{n\to\infty} d(ASx_n,SSx_n)=0 \text{ and } \lim_{n\to\infty} d(SAx_n,AAx_n)=0$$

Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Ax_n= \lim_{n\to\infty}Sx_n = t$ for some $t \in X$.

Definition 1.4[6]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be **A-compatible** if $\lim_{n\to\infty} d(ASx_n,SSx_n)=0$, Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \epsilon X$.

Definition 1.5[6]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be **S-compatible** if $\lim_{n\to\infty} d(SAx_n,AAx_n)=0$,Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \epsilon X$.

Definition 1.6[7]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be compatible of type (P) if $\lim_{n\to\infty} d(AAx_n,SSx_n)=0$, Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \epsilon X$.

Proposition 1.1[6]: Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is A-compatible on X and St = At for some $t \epsilon X$, then ASt= SSt.

Proposition 1.2[6]: Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is S-compatible on X and St = At for some $t \epsilon X$, then SAt= AAt.

Proposition 1.3[6]: Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is A-compatible on X and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \epsilon X$, then $SSx_n \to At$ if A is continuous at t.

Proposition 1.4[6]: Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is S-compatible on X and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \epsilon X$, then $AAx_n \to St$ if S is continuous at t.

The following result was proved by Branciari [2].

Theorem 1.1[2]: Let (X, d) be a complete metric space $c \epsilon]0,1[$ and let $f: X \to X$ be a mapping such that for each $x, y \epsilon X$,$\int_0^{d(fx,fy)} \varphi(t)dt \le c\int_0^{d(x,y)} \varphi(t)dt$, where $\varphi: [0,+\infty] \to [0,+\infty]$ is a Lebesgue integrable mapping which is summable i.e. with finite integral on each compact subset of $[0,+\infty]$ nonnegative and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$; then f has a unique fixed point $a \epsilon X$ such that for each $x \epsilon X$, $\lim_{n\to+\infty} f^n x = a$.

2. MAIN RESULT

In this paper, we prove fixed point theorem for compatible mapping of type (A), A-compatible and S-Compatible mappings satisfying contractive condition of integral type in a complete metric space. The result of this paper extends and the results of Aage and Salunke [1], Branciari [2], Murthy [5], Pathak and Khan [6], Shahidur Rahman et.al.[9],Sharma and Sahu[10]for generalized contraction of integral type was generalized.

Theorem 2.1: Let P, Q, S and T be self-maps of a complete metric space (X, d) satisfying the following conditions:

(i) $P(X) \subseteq S(X)$ and $Q(X) \subseteq T(X)$.

(ii) $\int_0^{d(px,Qy)} \varphi(t)dt \le \psi\{\int_0^{M(x,y)} \varphi(t)dt\}$, where $M(x,y)=d(Tx,Px)\{\frac{1+d(Sy,Qy)}{1+d(Tx,Sy)}\}$, for all $x, y \epsilon X$ and

(iii) $\varphi: [0,+\infty] \to [0,+\infty]$ is a Lebesgue integrable mapping which is summable i.e. with finite integral on each compact subset of $[0,+\infty]$ nonnegative and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$; also $\psi: R+ \to R+$ be a right continuous mapping Satisfying the condition $\psi(0)=0$ and $\psi(t) < t$ for each $t > 0$.

(iv) One of P, Q, S and T is continuous.

(v) Pairs (T, P) and (S, Q) are compatible mappings of type (A).

Then P, Q, S and T have unique common fixed point in X.

Proof: Let $x_0 \epsilon X$ be arbitrary. Choose a point x_1 in X such that $Px_0= Sx_1$. This can be done since $P(X) \subseteq S(X)$. Let x_2 be another point in X such that $Qx_1 = Tx_2$.This can be done since $Q(X) \subseteq T(X)$. In general, we can choose $x_{2n}, x_{2n+1}, x_{2n+2}$... Such that $Px_{2n}= Sx_{2n+1}$ and $Qx_{2n+1} =Tx_{2n+2}$. So that we obtain a sequence Px_0, Qx_1, Px_2, Qx_3 ... Hence in general, we define a sequence $\{y_{2n}\}$ in X as $y_{2n+1} = Px_{2n} = Sx_{2n+1}$;$y_{2n+2} = Qx_{2n+1} = Tx_{2n+2}$, where n = 0, 1, 2, 3,… .

Now we will show that the sequence $\{y_{2n}\}$ is Cauchy. For this put $x = x_{2n}$, $y = x_{2n+1}$ in (ii), we have $\int_0^{d(px_{2n},Qx_{2n+1})} \varphi(t)dt \leq \psi \int_0^{M(x,y)} \varphi(t)\, dt$,

Where $M(x,y)=d(Tx_{2n},Px_{2n})\{\frac{1+d(Sx_{2n+1},Qx_{2n+1})}{1+d(Tx_{2n},Sx_{2n+1})}\}$,

$$\int_0^{d(y_{2n+1},y_{2n+2})} \varphi(t)dt \leq \psi \int_0^{d(y_{2n},y_{2n+1})\{\frac{1+d(y_{2n+1},y_{2n+2})}{1+d(y_{2n},y_{2n+1})}} \varphi(t)\, dt ,$$

This gives $\int_0^{d(y_{2n+1},y_{2n+2})[1+d(y_{2n},y_{2n+1})]} \varphi(t)dt \leq \psi \int_0^{d(y_{2n},y_{2n+1})[1+d(y_{2n+1},y_{2n+2})]} \varphi(t)\, dt$,

This implies

$$\int_0^{d(y_{2n+1},y_{2n+2})} \varphi(t)dt + \int_0^{d(y_{2n+1},y_{2n+2})d(y_{2n},y_{2n+1})} \varphi(t)dt \leq \psi \int_0^{d(y_{2n},y_{2n+1})} \varphi(t)dt + \psi \int_0^{d(y_{2n},y_{2n+1})d(y_{2n+1},y_{2n+2})} \varphi(t)\, dt.$$

This gives $\int_0^{d(y_{2n+1},y_{2n+2})} \varphi(t)dt < \int_0^{d(y_{2n},y_{2n+1})} \varphi(t)dt$

Hence as a consequence; we have $\int_0^{d(y_{2n+1},y_{2n+2})} \varphi(t)dt \to 0 \; as \; n \to \infty.$

Hence the sequence $\{y_{2n}\}$ is Cauchy sequence in X.

Since $\{y_{2n}\}$ is Cauchy sequence and (X, d) is complete, so the sequence $\{y_{2n}\}$ has a limit point say z in X. Hence the sub sequences $\{Px_{2n}\} = \{Sx_{2n+1}\}$ and $\{Qx_{2n+1}\} = \{Tx_{2n+2}\}$ also converges to the point z in X. Suppose that the mapping T is continuous. Then $T^2x_{2n} \to Tz$ and $TPx_{2n} \to Tz$ as $n\to\infty$. Since the pair (T, P) is compatible of type (A). We get $PTx_{2n} \to Tz$

as $n\to\infty$. Now by (ii), if we put $x = Tx_{2n}$, $y = x_{2n+1}$, we get,

$$\int_0^{d(PTx_{2n},Qx_{2n+1})} \varphi(t)dt \leq \psi \int_0^{M(x,y)} \varphi(t)\, dt$$

Where $M(x, y) = d(TTx_{2n},PTx_{2n})\{ \frac{1+d(Sx_{2n+1},Qx_{2n+1})}{1+d(TTx_{2n},Sx_{2n+1})}\}$

Letting $n \to\infty$, we get $\int_0^{d(Tz,z)} \varphi(t)dt \leq \psi \int_0^{d(Tz,Tz)\{\frac{1+d(z,z)}{1+d(Tz,z)}\}} \varphi(t)\, dt$

This gives $\int_0^{d(Tz,z)} \varphi(t)dt \leq 0$.Hence Tz = z. Further, if we put x = z, $y = x_{2n+1}$ in (ii), we get

$$\int_0^{d(Pz,Qx_{2n+1})} \varphi(t)dt \leq \psi \int_0^{d(Tz,Pz)\{\frac{1+d(Sx_{2n+1},Qx_{2n+1})}{1+d(Tz,Sx_{2n+1})}\}} \varphi(t)\, dt$$

Letting $n\to\infty$, we get $\int_0^{d(Pz,z)} \varphi(t)dt \leq \psi \int_0^{d(z,Pz)\{\frac{1+d(z,z)}{1+d(z,z)}\}} \varphi(t)\, dt$

This gives $\int_0^{d(Pz,z)} \varphi(t)dt \leq \psi \int_0^{d(z,Pz)} \varphi(t)\, dt < \int_0^{d(z,Pz)} \varphi(t)\, dt$

Hence Pz = z .Thus Pz= Tz= z. Since P(X) ⊆ S (X), there is a point u ϵX such that z = Pz= Su.

Now by (ii), $\int_0^{d(z,Qu)} \varphi(t)dt = \int_0^{d(Pz,Qu)} \varphi(t)\, dt \leq \psi \int_0^{d(Tz,Pz)\{\frac{1+d(Su,Qu)}{1+d(Tz,Su)}\}} \varphi(t)\, dt$,

$$\int_0^{d(z,Qu)} \varphi(t)dt \leq \psi \int_0^{d(Tz,Tz)\{\frac{1+d(Su,Qu)}{1+d(Tz,Su)}\}} \varphi(t)\, dt$$

Hence , $\int_0^{d(z,Qu)} \varphi(t)dt \leq 0$. This gives Qu = z = Su. Take y_n= u for n ≥1. Then $Qy_n \to Qu$= z and $Sy_n \to Su$= z as $n \to\infty$. Since the pair (S, Q) is compatible of type (A), we get $\lim_{n\to\infty} d(QSy_n, SSy_n)=0$.Implies (Qz, Sz) = 0, since Sy_n= z for all

$n \geq 1$. Hence Qz= Sz. Finally, by (ii), we have $\int_0^{d(z,Qz)} \varphi(t)dt = \int_0^{d(Pz,Qz)} \varphi(t)\, dt \leq \psi \int_0^{d(Tz,Pz)\{\frac{1+d(Sz,Qz)}{1+d(Tz,Sz)}\}} \varphi(t)\, dt$

We get $\int_0^{d(z,Qz)} \varphi(t)dt \leq 0$. Hence z = Qz = Sz. Thus z = Pz= Tz=Qz = Sz.

Therefore z is common fixed point of P, T, Q and S, when the continuity of T is assumed. Now suppose that P is continuous. Then $P^2 x_{2n} \to Pz$, $PTx_{2n} \to Pz$ as $n \to \infty$. Since the pair (T, P) is compatible of type (A) therefore $TPx_{2n} \to Pz$ as $n \to \infty$. By condition (ii), we have

$$\int_0^{d(P^2x_{2n},Qx_{2n+1})} \varphi(t)dt = \psi \int_0^{d(PPx_{2n},Qx_{2n+1})} \varphi(t)dt \leq \psi \int_0^{d(TPx_{2n},PPx_{2n})\{\frac{1+d(Sx_{2n+1},Qx_{2n+1})}{1+d(TPx_{2n},Sx_{2n+1})}\}} \varphi(t)\, dt$$

Letting $n \to \infty$, we get $\int_0^{d(Pz,z)} \varphi(t)dt \leq \psi \int_0^{d(Pz,Pz)\{\frac{1+d(z,z)}{1+d(Pz,z)}\}} \varphi(t)\, dt$

We get, $\int_0^{d(Pz,z)} \varphi(t)dt \leq 0$. Hence Pz= z. But P(X) ⊆ S(X), there is a point v ϵX such that z = Pz= Sv. Now by (ii),

we have $\int_0^{d(P^2x_{2n},Qv)} \varphi(t)dt = \int_0^{d(PPx_{2n},Qv)} \varphi(t)dt \leq \psi \int_0^{d(TPx_{2n},PPx_{2n})\{\frac{1+d(Sv,Qv)}{1+d(TPx_{2n},Sv)}\}} \varphi(t)\, dt$

Letting $n \to \infty$ and using Pz = z, we get, $\int_0^{d(z,Qv)} \varphi(t)dt \leq \psi \int_0^{d(z,z)\{\frac{1+d(Sv,Qv)}{1+d(z,Sv)}\}} \varphi(t)\, dt$

This gives $\int_0^{d(z,Qv)} \varphi(t)dt \leq 0$. Hence Qv = z. Thus z = Sv = Qv for vϵX.

Let $y_n = v$. Then $Qy_n \to Qv = z$ and $Sy_n \to Qv = z$. Since (S, Q) is compatible of type (A), we have

$\lim_{n\to\infty} d(QSy_n, SSy_n)=0$,This gives QSv = SQv or Qz = Sz. Further by (ii), we have,

$$\int_0^{d(Px_{2n},Qz)} \varphi(t)dt \leq \psi \int_0^{d(Tx_{2n},Px_{2n})\{\frac{1+d(Sz,Qz)}{1+d(Tx_{2n},Sz)}\}} \varphi(t)\, dt$$

Letting $n \to \infty$ and using the results above, we get

$$\int_0^{d(z,Qz)} \varphi(t)dt \leq \psi \int_0^{d(z,z)\{\frac{1+d(Sz,Qz)}{1+d(z,Sz)}\}} \varphi(t)\, dt$$

This gives $\int_0^{d(z,Qz)} \varphi(t)dt \leq 0$

Thus z = Qz. Hence z = Qz = Sz. Since Q(X) ⊆ T(X), there is a point w ϵX such that z = Qz = Tw. Thus by (ii), we have

$$\int_0^{d(Pw,z)} \varphi(t)dt = \int_0^{d(Pw,Qz)} \varphi(t)dt \leq \psi \int_0^{d(Tw,Pw)\{\frac{1+d(Sz,Qz)}{1+d(Tw,Sz)}\}} \varphi(t)\, dt$$

$$\int_0^{d(Pw,z)} \varphi(t)dt \leq \psi \int_0^{d(z,Pw)\{\frac{1+d(z,z)}{1+d(z,z)}\}} \varphi(t)\, dt$$

$$\int_0^{d(Pw,z)} \varphi(t)dt \leq \psi \int_0^{d(z,Pw)} \varphi(t)\, dt < \int_0^{d(z,Pw)} \varphi(t)\, dt$$

This gives Pw = z. Take $y_n = w$ then $Py_n \to Pw = z$, $Ty_n \to Tw = z$. Since (T, P) is compatible of type (A), we get $\lim_{n\to\infty} d(PTy_n, TTy_n)=0$.This implies that PTw = TPw or Pz = Tz. Thus we have z = Pz = Tz = Sz = Qz. Hence z is a common fixed point of P, Q, S and T, when P is continuous. The proof is similar that z is common fixed point of P, Q, S and T, when the continuity of S or Q is assumed.

Uniqueness: Let z and t be two common fixed point of P, Q, S and T.

i.e. z = Pz = Tz = Qz = Sz and t = Pt = Tt = Qt = St. By condition (ii), we have

$$\int_0^{d(z,t)} \varphi(t)dt = \int_0^{d(Pz,Qt)} \varphi(t)dt \leq \psi \int_0^{d(Tz,Pz)\{\frac{1+d(St,Qt)}{1+d(Tz,St)}\}} \varphi(t)dt$$

This gives $\int_0^{d(z,t)} \varphi(t)dt \leq 0$.Thus we have z = t. Hence z is a unique common fixed point of mappings P, Q, S and T.

Theorem 2.2: Let P, Q, S and T be self maps of a complete metric space (X, d) satisfying the conditions (i), (ii), (iv) and ψ be as in theorem 2.1 satisfying the inequality

$$d(Px, Qy) \leq \psi\, d(Tx, Px)\{\frac{1+d(Sy,Qy)}{1+d(Tx,Sy)}$$

Then P, Q, S and T have unique common fixed point in X.

Proof: The proof of the theorem 2.2 is follows from theorem 2.1 by putting φ (t)=1 in (ii).

Corollary 2.1: Let P, Q, S and T be self-maps of a complete metric space (X, d) satisfying the following conditions (i),(ii),(iii) of theorem 2.1and if pairs (T, P) and (S, Q) are A- compatible or S- compatible. Then P, Q, S and T have unique common fixed point in X.

Proof: The proof of the corollary directly follows by splitting the definition of compatible mappings of type (A) into A-compatible or S- compatible mappings and using the Proposition 1.1 to 1.4.

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