

Global Convergence properties of A Modified HS Conjugate Gradient Methods for Unconstrained optimization

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ABSTRACT

A modified HS conjugate gradient method for unconstrained optimization problems is suggested and analyzed, for any inexact line search its satisfies the descent condition and the global convergence of the proposed method is proved by using Wolf conditions . Numerical results indicate the new method is attractive for the given test problems.

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1. INTRODUCTION

A nonlinear conjugate gradient method (CG) is powerful technique for solving large scale unconstrained optimization problems :

$$\min\{f(x): x \in R^n\} \quad \dots \dots \dots (1)$$

where $f: R^n \rightarrow R$ is a continuously differentiable function [8]. The method is iterative beginning by an initial point $x_0 \in R^n$, they generate a sequence of points $\{x_i \in R^n\}$ by the procedure :

$$x_{i+1} = x_i + \alpha_i d_i \quad i = 0, 1, 2, \dots \dots \dots (2)$$

where the step size α_i is computed by the line search $\alpha_i > 0$ and satisfies the Wolf conditions (W.C.) :

$$f(x_i + \alpha_i d_i) - f(x_i) \leq \delta \alpha_i g_i^T d_i \quad \dots \dots \dots (3)$$

$$|g_{i+1}^T d_i| \leq -\sigma g_i^T d_i \quad \dots \dots \dots (4)$$

with $0 < \delta < \sigma < 1$, and $f_i = f(x_i)$, $g_i = g(x_i)$, g_i is the gradient of f evaluated at the current iterate x_i .[11] and d_i is the search direction computed by

$$d_i = \begin{cases} -g_i & \text{for } i = 1 \\ -g_i + \beta_i d_{i-1} & \text{for } i \geq 2 \end{cases} \quad \dots \dots \dots \dots \dots (5)$$

where g_i denotes $\nabla f(x)$ and β_i is parameter well-known as the coefficient conjugate gradient well known as standard formulas for β_i given by :

$$\beta_i^{HS} = \frac{g_i^T y_{i-1}}{d_{i-1}^T y_{i-1}} \quad [6] \quad \dots \dots \dots \dots \dots (6)$$

$$\beta_i^{PR} = \frac{g_i^T y_{i-1}}{\|g_{i-1}\|^2} \quad [9] \quad \dots \dots \dots \dots \dots (7)$$

$$\beta_i^{FR} = \frac{\|g_i\|^2}{\|g_{i-1}\|^2} \quad [5] \quad \dots \dots \dots \dots \dots (8)$$

where g_i and g_{i-1} are the gradients of (x). [10]

Ibrahim .M.A, Mamat .M. and Leong. W. [7] suggested a search direction will be implemented into anew search known as the HBFGS method and given by :

$$d_i = \begin{cases} -H_i^{-1}g_i & i = 0 \\ -H_i^{-1}g_i + \lambda d_{i-1} & i \geq 0 \end{cases} \quad \dots \dots \dots (9)$$

where H_i is the BFGS updating matrix by the BFGS formula [12]

$$H_{i+1} = H_i - \frac{H_i s_i s_i^T H_i}{s_i^T H_i s_i} + \frac{y_i y_i^T}{s_i^T y_i} \quad \dots \dots \dots (10)$$

which satisfied the QN condition :

$$H_{i+1}y_i = s_i \quad \dots \dots \dots (11)$$

where :

$$\begin{aligned} y_i &= g_{i+1} - g_i \\ s_i &= x_{i+1} - x_i \end{aligned}$$

2. NEW BETA OF CONJUGATE GRADIENT METHOD β^{ES}

In this section we present a modified of a new beta

from eq.(9) we have

$$d_i = -H_i^{-1}g_i + \lambda_i d_{i-1} \quad \dots \dots \dots (12)$$

and the CG direction

$$d_i = -g_i + \beta_i d_{i-1} \quad \dots \dots \dots (13)$$

by equality (12) & (13)

$$-H_i^{-1}g_i + \lambda_i d_{i-1} = -g_i + \beta_i d_{i-1} \quad \dots \dots \dots (14)$$

Multiply both said of eq.(14) by y_{i-1}

$$\beta_i d_{i-1}^T y_{i-1} = g_i^T y_{i-1} - (H_i^{-1}g_i)^T y_{i-1} + \lambda_i d_{i-1}^T y_{i-1} \quad \dots \dots \dots (15)$$

since from QN method [1]

$$(-H_i^{-1}g_i)^T y_{i-1} = -g_i^T (H_i^{-1}y_{i-1})$$

from eq.(11) we get :

$$= -g_i^T s_{i-1} \quad \dots \dots \dots (16)$$

$$\beta_i d_{i-1}^T y_{i-1} = g_i^T y_{i-1} - g_i^T s_{i-1} + \lambda_i d_{i-1}^T y_{i-1} \quad \dots \dots \dots (17)$$

$$\beta_i = \frac{g_i^T y_{i-1}}{d_{i-1}^T y_{i-1}} - \frac{g_i^T s_{i-1}}{d_{i-1}^T s_{i-1}} + \lambda_i \quad \dots \dots \dots (18)$$

where λ_i is defined by Al Naemi – Ghada [1]

$$\lambda_i = \frac{-g_i^T d_{i-1}}{d_{i-1}^T g_{i-1}} \quad \dots \dots \dots (19)$$

then the new β_i^{ES} is defined by :

$$\beta_i^{ES} = \frac{g_i^T y_{i-1}}{d_{i-1}^T y_{i-1}} - \frac{g_i^T s_{i-1}}{d_{i-1}^T s_{i-1}} - \frac{g_i^T d_{i-1}}{d_{i-1}^T g_{i-1}} \quad \dots \dots \dots (20)$$

If we use EIS then new beta β_i^{ES} is become β_i^{HS} .

and then the search direction is defined by

$$d_i = -g_i + \left(\frac{g_i^T y_{i-1}}{d_{i-1}^T y_{i-1}} - \frac{g_i^T s_{i-1}}{d_{i-1}^T s_{i-1}} - \frac{g_i^T d_{i-1}}{d_{i-1}^T g_{i-1}} \right) d_{i-1} \quad \dots \dots \dots (21)$$

3. ALGORITHM

Step (1) : put the initial point $x_1 \in R$ and compute $f(x_1)$, $g(x_1)$, $d_1 = -g_1$ and $i=1$

Step (2) : the test criterion for stopping the iterations if satisfied

$\|g_i\| \leq \epsilon$, $\epsilon = 10^{-7}$ then stop x_i is the optimal solution

else go to step (3)

Step (3) : determine the step length α_i by using Wolfe line search condition eq.(3 & 4)

and update variable $x_{i+1} = x_i + \alpha_i d_i$,

Step (4) : calculate search direction d_i by eq.(21), where β^{ES} is defined by eq.(20).

Step (5) : calculate f_{i+1} , g_{i+1} , y_i , s_i .

Step (6) : set $i=i+1$ and go to step (2)

4. CONVERGENCE

We offer global convergence properties of β^{ES} so that we presume the search direction d_i satisfied the descent condition . for all $i \geq 0$, there exists a constant $c_1 > 0$ such that

$$\frac{g_i^T d_i}{\|g_i\|^2} \leq -c_1 \text{ for all } i \geq 0 \dots \dots \dots \quad (22)$$

then the search directions satisfy the sufficient descent condition so we need to make a few assumption based on the objective function .

Assumption 4.1:[3]

- A- The level set $S = \{x \in R^n, f(x) \leq f(x_0)\}$ is bounded and convex .
- B- In some neighborhood N of S , f is continuously differentiable and its gradient is lipschitz continuous. that is to say there exists a constant $l > 0$ such that :
 $\|g(x) - g(y)\| \leq l\|x - y\| \quad , \forall x, y \in N \dots \dots \dots \quad (23)$

clearly assumption 4.1 imply that there exists a constant γ such that:

$$\|g(x)\| \leq \gamma \quad , \forall x \in N \dots \dots \dots \quad (24)$$

Lemma 4.2 : [4]

Since f is a uniformly convex function on S , there exists a constant $\mu > 0$, Such that :

$$(\nabla f(x) - \nabla f(y))^T (x - y) \geq \mu\|x - y\|^2 , \quad \forall x, y \in S \dots \dots \dots \quad (25)$$

from eq.(23) ,(25) we have

$$\mu\|s_i\|^2 \leq y_i^T s_i \leq l\|s_i\|^2 \dots \dots \dots \quad (26)$$

Theorem 4.3:

The new search directions d_i

$$d_i = -g_i + \left(\frac{g_i^T y_{i-1}}{d_{i-1}^T y_{i-1}} - \frac{g_i^T s_{i-1}}{d_{i-1}^T y_{i-1}} + \frac{-g_i^T d_{i-1}}{d_{i-1}^T g_{i-1}} \right) d_{i-1}$$

satisfy the sufficient descent condition eq.(22)

Proof:

since $d_i = -g_i + \beta_i d_{i-1}$

$$\begin{aligned} g_i^T d_i &= -g_i^T g_i + g_i^T \beta_i d_{i-1} \\ &= -\|g_i\|^2 + \left(\frac{g_i^T y_{i-1} - g_i^T s_{i-1}}{d_{i-1}^T y_{i-1}} - \frac{g_i^T d_{i-1}}{d_{i-1}^T g_{i-1}} \right) g_i^T d_{i-1} \end{aligned}$$

from Wolfe condition eq.(4)

$$\begin{aligned} g_{i+1}^T d_i &\leq -\sigma g_i^T d_i \\ g_i^T d_i &\leq -\|g_i\|^2 + \left(\frac{g_i^T (g_i - g_{i-1}) - g_i^T (\alpha d_{i-1})}{d_{i-1}^T y_{i-1}} + \frac{\sigma g_{i-1}^T d_{i-1}}{d_{i-1}^T g_{i-1}} \right) g_i^T d_{i-1} \end{aligned}$$

from Powell restart [9]

$$|g_{i+1}^T g_i| \geq 0.2 \|g_{i+1}\|^2$$

$$-g_{i+1}^T g_i \leq -0.2 \|g_{i+1}\|^2$$

$$g_{i+1}^T d_i = d_i^T y_i + g_i^T d_i$$

$$g_{i+1}^T d_i < d_i^T y_i$$

$$\begin{aligned} &\leq -\|g_i\|^2 + \left(\frac{\|g_i\|^2 - 0.2 \|g_i\|^2 - \alpha d_{i-1}^T y_{i-1}}{d_{i-1}^T y_{i-1}} + \sigma \right) d_{i-1}^T y_{i-1} \\ &\leq -\|g_i\|^2 + (0.8\|g_i\|^2 - (\alpha - \sigma)d_{i-1}^T y_{i-1}) \end{aligned}$$

if $\alpha > \sigma$ then

$$\leq -\|g_i\|^2 + 0.8\|g_i\|^2$$

$$\leq -0.2\|g_i\|^2$$

when $c=0.2$

$$\therefore g_i^T d_i \leq -c\|g_i\|^2$$

Theorem 4:4 (Global Convergence)

Suppose that the new search directions d_i defined by eq.(21) satisfy the sufficient descent condition and Assumption 4.1 hold then

$$\lim_{i \rightarrow \infty} \|g_i\|^2 = 0$$

Proof:

$$\begin{aligned} d_i &= -g_i + B_i d_{i-1} \\ \|d_i\| &\leq \|g_i\| + \|B_i d_{i-1}\| \\ &\leq \|g_i\| + \left\| \left(\frac{g_i^T y_{i-1} - g_i^T s_{i-1}}{d_{i-1}^T y_{i-1}} - \frac{g_i^T d_{i-1}}{d_{i-1}^T g_{i-1}} \right) d_{i-1} \right\| \end{aligned}$$

Let :

$$\begin{aligned} g_i^T y_{i-1} &= y_{i-1}^T g_i \\ g_i^T s_{i-1} &= s_{i-1}^T g_i \\ g_i^T d_{i-1} &= d_{i-1}^T g_i \\ &\leq \|g_i\| + \left| \left(\frac{y_{i-1}^T g_i - s_{i-1}^T g_i}{d_{i-1}^T y_{i-1}} - \frac{d_{i-1}^T g_i}{d_{i-1}^T g_{i-1}} \right) \right| \|d_{i-1}\| \\ &\leq \|g_i\| + \left(\frac{\|y_{i-1}\| \|g_i\| - \|s_{i-1}\| \|g_i\|}{\|d_{i-1}\| \|y_{i-1}\|} - \frac{\|d_{i-1}\| \|g_i\|}{\|d_{i-1}\| \|g_{i-1}\|} \right) \|d_{i-1}\| \\ &\leq \|g_i\| + \|g_i\| \left(\frac{\|y_{i-1}\| \|d_{i-1}\| - \|s_{i-1}\| \|d_{i-1}\|}{\|d_{i-1}\| \|y_{i-1}\|} - \frac{\|d_{i-1}\| \|d_{i-1}\|}{\|d_{i-1}\| \|g_{i-1}\|} \right) \end{aligned}$$

since $y \leq \alpha l \|d\|$

$$y^T d \leq \alpha l \|d\|^2$$

$$\mu s \leq y$$

$$\mu \alpha \|d\|^2 \leq y^T d$$

from eq.(26) we have

$$\begin{aligned} \mu \alpha \|d\|^2 &\leq y^T d \leq \alpha l \|d\|^2 \\ \|d_i\| &\leq \|g_i\| + \|g_i\| \left(\frac{\alpha l \|d_{i-1}\|^2 - \alpha \|d_{i-1}\|^2}{\mu \alpha \|d_{i-1}\|^2} - \frac{\|d_{i-1}\|}{\|g_{i-1}\|} \right) \\ &\leq \|g_i\| + \|g_i\| \left(\frac{l-1}{\mu} - \frac{\|d_{i-1}\|}{\|g_{i-1}\|} \right) \\ &\leq \|g_i\| + \frac{l}{\mu} \|g_i\| - \left(\frac{1}{\mu} + \frac{\|d_{i-1}\|}{\|g_{i-1}\|} \right) \|g_i\| \end{aligned}$$

since $\left(\frac{1}{\mu} + \frac{\|d_{i-1}\|}{\|g_{i-1}\|} \right) < 0$

$$\|d_i\| \leq \left(1 + \frac{l}{\mu} \right) \|g_i\|$$

$$\text{let } c = \left(1 + \frac{l}{\mu} \right)$$

$$\begin{aligned} \therefore \|d_i\| &\leq c \|g_i\| \\ \|d_i\|^2 &\leq c \|g_i\|^2 = \varphi^2 \\ \sum_{i=1}^{\infty} \frac{1}{\|d_i\|^2} &\geq \frac{1}{\varphi^2} \sum_{i=1}^{\infty} 1 = \infty \\ \lim_{i \rightarrow \infty} \|g_i\|^2 &= 0 \end{aligned}$$

5. NUMERICAL RESULT

We offer some numerical result, also we use the test functions are the stander unconstrained optimization problem from Andrei [3], We analysis the performance of the HS method with the new algorithm, Table(1) is the comparisons between the Hestenes-Staifel method and New method. such that the dimensions of the tests range between 1000 and 10000 and Table(2) is the comparisons between the Hestenes-Staifel method and New method for the total of all dimensions of the tests range of between 1000 ,2000, 10000. the comparative performance of the algorithm is evaluated by considering both the (NOF) which is the number of function evaluations and the (NOI) which is the number of iterations and all these methods terminate when the following stooping criterion

$$\|g_i\| \leq \epsilon, \epsilon = 10^{-7}$$

Table (1)Numerical Comparisons between the Hestenes-Staifel method and New method.

N	Test fu.	Dim	HS	New
1	Trigonometric	1000	68(5)	14(4)
		10000	38(5)	13(4)
2	White & holst	1000	32(5)	36(3)
		10000	29(7)	33(6)
3	Beale	1000	10(2)	10(2)
		10000	12(2)	11(2)
4	Penalty	1000	2(1)	2(1)
		10000	15(1)	2(1)
5	Raydon 2	1000	4(2)	4(1)
		10000	4(1)	4(1)
6	Generalized tridiag	1000	50(2)	40(1)
		10000	56(3)	61(2)
7	Diagonal 4	1000	4(1)	4(1)
		10000	4(1)	6(1)
8	extended psc1	1000	9(1)	7(2)
		10000	9(1)	8(2)
9	Exteaded Himmeblau	1000	7(1)	6(1)
		10000	7(1)	6(1)
10	Extended powel 1	1000	207(2)	67(4)
		10000	132(1)	113(3)
11	Extended BD1	1000	38(4)	34(10)
		10000	27(3)	31(15)
12	Extended quadratic penalty QP1	1000	12(2)	26(1)
		10000	235(27)	24(2)
13	Extended quadratic penalty QP2	1000	35(2)	2(2)
		10000	29(2)	2(1)
14	Dixmaan B cute	1000	31(1)	15(4)
		10000	24(5)	21(4)
15	Rosenbrock	1000	26(1)	29(2)
		10000	28(2)	25(2)

Table (2) The total Numerical Comparisons between the Hestenes-staifel method and New method.

N	Test fu.	HS	New
1	Trigonometric	381(25)	199(39)
2	White & holst	315(30)	324(37)
3	Beale	113(18)	111(21)
4	Penalty	71(16)	44(22)
5	Raydon 2	40(20)	40(10)
6	Generalized tridiag	555(10)	580(18)
7	Diagonal 4	40(10)	48(10)
8	extended psc1	5752(50)	5835(95)
9	Exteaded Himmeblau	70(3)	60(10)
10	Extended powel 1	990(5)	955(29)
11	Extended BD1	398(49)	320(114)
12	Extended quadratic penalty QP1	433(65)	265(26)
13	Extended quadratic penalty QP2	292(29)	20(16)
14	Dixmaan B cute	212(14)	182(32)
15	Rosenbrock	260(20)	270(27)

CONCOLUSION

In this research, we obtained a new algorithm, we proved the condition of generate sufficient descent directions as well as the achievement of globally convergent . it showed the numerical results that the new algorithm at the level of efficiency to solve the general functions and also proved to be efficient when it relied on NOF and NOI.

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Appendix

To know the specifics test functions that used in search can be Look in [3]:

1- Trigonometric function :

$$f(x) = \sum_{i=1}^n \left[\left(n - \sum_{j=1}^n \cos x_j \right) + i(1 - \cos x_i) - \sin x_i \right]^2$$

$$x_0 = [0.2, 0.2, \dots, 0.2]^T$$

2- White & holst function :

$$f(x) = \sum_{i=1}^{n/2} c(x_{2i} - x_{2i-1}^3)^2 + (1 - x_{2i-2})^2$$

$$x_0 = [-1.2, 1, \dots, -1.2, 1]^T \quad c = 100$$

3- Beale Function,

$$F(x) = (1.5 - x_1(1 - x_2))^2 + (2.25 - x_1(1 - x_2^2))^2 + (2.625 - x_1(1 - x_2^3))^2,$$

$$x_0 = (0, 0)^T$$

4- Penalty function :

$$f(x) = \sum_{i=1}^{n-1} (x_i - 1)^2 + \left(\sum_{j=1}^n x_j^2 - 0.25 \right)^2$$

$$x_0 = [1, 2, \dots, n]^T$$

5- Raydan2 Function

$$f(x) = \sum_{i=1}^n (\exp(-x_i) - x_i),$$

$$x_0 = [1, 1, \dots, 1]^T.$$

6- eneralization tridiag function :

$$f(x) = \sum_{i=1}^{n-1} (x_i + x_{i+1} - 3)^2 + (x_i - x_{i-1} + 1)^2$$

$$x_0 = [2, 2, \dots, 2]^T$$

7- Diagonal-4 Function:

$$f(x) = \sum_{i=1}^{n/2} \frac{1}{2} (x_{2i-1}^2 + c x_{2i}^2),$$

$$x_0 = [1, 1, \dots, 1]^T, c = 100$$

8- Extended Psc1 Function:

$$f(x) = \sum_{i=1}^{n/2} (x_{2i-1}^2 + x_{2i}^2 + x_{2i-1}x_{2i})^2 + \sin^2(x_{2i-1}) + \cos^2(x_{2i}),$$

$$x_0 = [3, 0.1, \dots, 3, 0.1]^T$$

9- Extended Himmelblau Function:

$$f(x) = \sum_{i=1}^{n/2} (x_{2i-1}^2 + x_{2i}^2 - 11)^2 + (x_{2i-1}^2 + x_{2i}^2 - 7)^2,$$

$$x_0 = [1, 1, \dots, 1]^T$$

10- Extended Powell Function:

$$f = \sum_{i=1}^{n/4} [x_{4i-3} + 10 x_{4i-2}]^2 + 5 (x_{4i-1} - x_{4i})^2 + (x_{4i-2} - 2 x_{4i-1})^4 + 10 (x_{4i-3} - x_{4i})^4 ; \quad x_0 = (3, -1, 0, 1, \dots)^T$$

11- Extended BD1 function:

$$f(x) = \sum_{i=1}^{n/2} (x_{2i-1}^2 + x_{2i}^2 - 2)^2 + (\exp(2x_{2i-1} - 1) - x_{2i})^2$$

$$x_0 = [0.1, 0.1, \dots, 0.1]^T$$

12- Extended quadratic penalty QP1 function :

$$f(x) = \sum_{i=1}^{n-1} (x_i^2 - 2)^2 + \left(\sum_{j=1}^n x_j^2 - 0.5 \right)^2$$

$$x_0 = [1, 1, \dots, 1]^T$$

13- Extended quadratic penalty QP2 function :

$$f(x) = \sum_{i=1}^{n-1} (x_i^2 - \sin x_i)^2 + \left(\sum_{j=1}^n x_j^2 - 100 \right)^2$$

$$x_0 = [1, 1, \dots, 1]^T$$

14- Dixmaan B Function (Cute):

$$f(x) = 1 + \sum_{i=1}^n \alpha x_i^2 \left(\frac{i}{n} \right)^{k1} + \sum_{i=1}^{n-1} \beta x_i^2 (x_{i+1} + x_{i+1}^2)^2 \left(\frac{i}{n} \right)^{k2} + \sum_{i=1}^{2m} \gamma x_i^2 x_{i+m}^4 \left(\frac{i}{n} \right)^{k3}$$

$$+ \sum_{i=1}^m \delta x_i x_{i+2m} \left(\frac{i}{n} \right)^{k4},$$

$$m = n/3 ; \alpha = 1; \beta = 0.0625 ; \gamma = 0.0625 ; \delta = 0.0625 ; k1 = 0, k2 = 0, k3 = 0, k4 = 1$$

$$x_0 = [2, 2, \dots, 2]^T$$

15- Rosenbrock function :

$$f(x) = \sum_{i=1}^{n/2} c(x_{2i} - x_{2i-1}^2)^2 + (1 - x_{2i-1})^2$$

$$x_0 = [-1.2, 1, \dots, -1.2, 1]^T \quad c = 100$$