On Equitable Coloring Parameters of Certain Graphs

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ABSTRACT

A proper coloring colors to the vertices of G such that the numbers of vertices in any two color classes differ by at most one, is called an equitable colouring of G. In this paper, we study two statistical parameters of five different graphs and we apply equitable coloring for these graphs. Here the graphs are Tadpole, windmill, friendship, gear and sunlet.

Keywords: Equitable coloring; coloring mean; coloring variance; χ_e – Chromatic mean; χ_e– Chromatic variance, Tadpole graph($T_{(m,n)}$), Windmill graph($W_d(k,\ n)$), Friendship graph(F_n), Gear graph(G_n), Sunlet graph(S_n).

INTRODUCTION

In this paper all the graphs we are considering are simple, finite, connected and undirected.

Graph coloring is an assignment of colors or labels or weights to the vertices, edges and faces of a graph under consideration. Many practical and real life situations motivated in the development of different types of graph coloring problems. Unless stated otherwise, the graph coloring is meant to be an assignment of colors to the vertices of a graph subject to certain conditions. A *proper coloring* of a graph G is a colouring with respect to which vertices of G are colored in such a way that no two adjacent vertices G have the same color.

Coloring of the vertices of a given graph G can be considered as a random experiment. For a proper k-coloring $c_1,c_2,c_3,\ldots\ldots,c_k\}$ of G, we can define a random variable (r.v) X which denotes the color (or precisely, the subscript i of the color c_i) of any arbitrary vertex in

G. As the sum of all weights of colors of G is equal to the number of vertices of G, the real valued function f(i) defined by,

$$f(i) = \begin{cases} \dfrac{\theta(c_i)}{\{|V(G)|}; & i = 1,2,\ldots\ldots,k \\ 0 & ; \quad elsewhere \end{cases}$$

will be the probability mass function (p.m.f.) of the random variable (r.v.) X, (see [7]),where $\theta(c_i)$ denotes the cardinality of the colour class of the colour c_i.

If the context is clear, we can also say that f(i) is the probability mass function of the graph G with respect to the given coloring, we can also define the parameters like mean and variance of the random variable X, with respect to a general coloring of G can be defined as follows.

An equitable coloring of G is a proper coloring which an assignment of colors to the vertices of G such that the numbers of vertices in any two color classes differ by at most one. The equitable chromatic number of a graph G is the smallest number k such that G has an equitable coloring with k colors.

Throughout the paper, we follow the convention that 0≤ $\theta(c_i)$ - $\theta(c_j)$ ≤ 1, when i < j.

PRELIMINARIES

Two types of chromatic means and variances corresponding to an equitable coloring of a graph G are defined as follows.

[8] Let, $c_2, c_3, \ldots\ldots, c_k\}$ be an equitable k-coloring of a given graph G and X be the random variable which denotes the number of vertices having a particular color in hthe p.m.f $f(i)$. Then,

(i) The equitable coloring mean of a coloring given graph G, denoted by μ_{χ_e} (G), isdefined to be to be μ_{χ_e} (G) = $\sum_{i=1} i\, f(i)$;

(ii) Equitable coloring variance of a coloring on a given graph G, denoted by $\sigma^2_{\chi_e}(G)$, is defined to be $\sigma^2_{\chi_e}(G) = \sum_{i=1} i^2 f(i) - \left(\sum_{i=1} i\, f(i)\right)^2$.

If the context is clear, the above defined parameters can respectively be called the equitable coloring mean and equitable coloring variance of the graph G with respect to the coloring In this paper we study the equitable coloring parameters of certain graphs such asTadpole, Windmill, Friendship, Gear and Sunlet.

1) *Definitions:*

(i) Tadpole Graph [11]

A tadpole graph $T_{(m,n)}$ is a graph obtained by joining a cycle C_m, m≥ 3 to a path P_n. n ≥ 1 with a bridge.

(ii) *Windmill Graph [12]*

The windmill graph W_d(k,n) is an undirected graph constructed for k ≥ 2, n≥ 2 by joining n copies of the complete graph $K_k$at a shared universal vertex. That is a 1-clique sumof these complete graphs.

(i) *Friendship Graph [13]*

A friendship graph F_n is a planar undirected graph with 2n+1 vertices and 3n edges. It can be constructed by joining n copies of the " cycle graph" C_3 with a common vertex.

(ii) *Gear Graph [14]*

A gear graph denoted G_n is a wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle. ThusG_n has 2n+1 vertices and 3n edges.

(iii) *Sunlet Graph [15]*

The n-sunlet graph of 2n vertices is obtained by attaching n pendant edges to a cycle$C_n$and is denoted by S_n. Sunlet graphs are also been called as "crown graph".

Theorem 1. For a Tadpole graph $T_{(3,n)}$, we have

$$\mu_{\chi_e}(T_{(3,n)}) = \begin{cases} \frac{n^2+2n+4}{4n}, & if\ n\ is\ even \\ \frac{n^2+2n+1}{4n}, & if\ n\ is\ odd \end{cases}$$

and

$$\sigma^2_{\chi_e}(T_{(3,n)}) = \begin{cases} \frac{n^4+20n^2-48}{48n^2}, & if\ n\ is\ even \\ \frac{n^4+2n^2-3}{48n^2}, & if\ n\ is\ odd \end{cases}$$

Proof. Letv_1, v_2, v_3,………v_n be the vertices of a tadpole graph $T_{(3,n)}$. Here 3 represents the total number of vertices on a cycle. n denotes the number of vertices connected by a path. Hence, every other colour class, with respect to an equitable colouring of $T_{(3,n)}$ must be a singleton (or) a 2-element set. Hence we need to consider the following cases.

Case1: Let n be even. Then $^{n-2} + 2$ colours are required in an equitable colouring of T .

Let $\mathscr{C} = \{c_1, c_2, c_3, \ldots\ldots, c_k, c_{k+1}, c_{k+2}\}$ *be an* equitable colouring of $T_{(3,n)}$, where $k = \frac{n-2}{2}$.

Note that the colour classes of $c_1, c_2, c_3, \ldots\ldots, c_k$ are 2-element sets. While the colour classes of c_{k+1} and c_{k+2} are singleton sets. Let X be the random variable which represents the colour of an arbitrarily chosen vertex of $T_{(3,n)}$. Then the corresponding p.m.f of G is,

$$f(i) = p(X = i) = \begin{cases} \frac{2}{n}, & for\ i = 1,2,\ldots\ldots.\frac{n-2}{2} \\ \frac{1}{n}, & for\ i = \frac{n-2}{2}+1, \frac{n-2}{2}+2 \\ 0, & elsewhere \end{cases}$$

Hence, the corresponding equitable colouring mean of the tadpole graph $T_{(3,n)}$ is given by,

$$\mu_{\chi_e}(T_{(3,n)}) = \left(1+2+3+\cdots+\frac{n-2}{2}\right).\frac{2}{n} + \left(\frac{n-2}{2}+1+\frac{n-2}{2}+2\right).\frac{1}{n} = \frac{n^2+2n+4}{4n}$$ *and* the

Corresponding equitable colouring variance of $T_{(3,n)}$ is,

$$\sigma^2_{\chi_e}(T_{(3,n)}) = \left(1^2+2^2+3^2+\cdots+\frac{(n-2)^2}{4}\right).\frac{2}{n} + \left[\left(\frac{n-2}{2}+1\right)^2+\left(\frac{n-2}{2}+2\right)^2\right].\frac{1}{n} - \left(\frac{n^2+2n+4}{4n}\right)^2 = \frac{n^4+20n^2-48}{48n^2}.$$

Case 2. Let n be odd. Then, $\frac{n-1}{2}+1$ colours are required in an equitable colouring of $T_{(3,n)}$.

If $\{c_1, c_2, c_3, \ldots, c_k, c_{k+1}\}$ be an equitable colouring of $T_{(3,n)}$. Where $k = \frac{n-1}{2}$. Note that the colour classes of $c_1, c_2, c_3, \ldots\ldots, c_k$ are 2-element sets, while the colour class of c_{k+1} is a singleton set. Let X be the random variable which represents the colour of an arbitrarily chosen vertex of $T_{(3,n)}$. Then the corresponding p.m.f of G is,

$$f(i) = p(X = i) = \begin{cases} \frac{2}{n}, & for\ i = 1,2,\ldots\ldots \frac{n-1}{2} \\ \frac{1}{n}, & for\ i = \frac{n-1}{2}+1 \\ 0, & elsewhere \end{cases}$$

Then, the corresponding equitable colouring mean is,

$$\mu_{\chi e}(T(3,n)) = \left(1 + 2 + 3 + \cdots + \frac{n-1}{2}\right).\frac{2}{n} + \left(\frac{n-1}{2} + 1\right).\frac{1}{n} = \frac{n^2+2n+1}{4n}$$ and

$$\sigma^2_{\chi e}(T(3,n)) = \left(1^2 + 2^2 + 3^2 + \cdots + \frac{(n-1)^2}{4}\right).\frac{2}{n} + \left(\left(\frac{n-1}{2}+1\right)^2\right).\frac{1}{n} - \left(\frac{n^2+2n+1}{4n}\right)^2 = \frac{n^4+2n^2-3}{48n^2}.$$

Theorem 2. For a windmill graph $W_d(4, n) = C_n+L_1$, we have

$$\mu_{\chi e}(W_d(4, n)) = \begin{cases} \frac{(n+2)^2}{4(n+1)}, & if\ n\ is\ even \\ \frac{n^2+4n+19}{4(n+1)}, & if\ n\ is\ odd \end{cases}$$

$$\sigma^2_{\chi e}(W_d(4,n)) = \begin{cases} \frac{n^4+4n^3+8n^2+8n}{48(n+1)^2}, & if\ n\ is\ even \end{cases}$$

Proof. Let v be the central vertex of the windmill graph $W_d(4, n) = C_n+L_1$ and $v_1, v_2, \ldots v_n$ be the vertices of the cycle C_n. Since v is adjacent to all other vertices of W_d, none of them can have the same colour of v. Hence, every other colour class, with respect to an equitable colouring of W_d must be a singleton (or) a 2-element set. Here we need to consider the following cases.

Case 1. Let n be even, Then $\frac{n}{2} + 1$ *colours* are required in an equitable colouring of W_d. Let

$\mathcal{C} = \{c_1, c_2, \ldots\ldots, c_m, c_{m+1}\}$ be an equitable colouring of W_d, where m = $\frac{n}{2}$. Note that the colour classes of $c_1, c_2, c_3, \ldots, c_m$ are 2-element sets, while the colour class of c_{m+1}, *which consists of the* central vertex is a singleton set. Let X be the random variable which represents the colour of an arbitrarily chosen vertex of $W_d(4, n)$. Then, the corresponding p.m.f of G is,

$$f(i) = p(X = i) = \begin{cases} \frac{2}{n+1}, & for\ i = 1,2,\ldots\ldots \frac{n}{2} \\ \frac{1}{n+1}, & for\ i = \frac{n}{2} + 1, \\ 0, & elsewhere \end{cases}$$

Hence, the corresponding equitable colouring mean of the windmill graph $W_d(4, n)$ is givenby,

$$\mu_{\chi e}(W_d) = \left(1 + 2 + 3 + \cdots + \frac{n}{2}\right) \cdot \frac{2}{n+1} + \left(\frac{n}{2} + 1\right) \cdot \frac{1}{n+1} = \frac{(n+2)^2}{4(n+1)}$$

and the corresponding variance of $W_d(4, n)$ is,

$$\sigma^2_{\chi e}(W_d) = \left(1^2 + 2^2 + 3^2 + \cdots + \left(\frac{n}{2}\right)^2\right) \cdot \frac{2}{n+1} + \left(\left(\frac{n}{2} + 1\right)^2\right) \cdot \frac{1}{n+1} - \left(\frac{(n+2)^2}{4(n+1)}\right)^2 = \frac{n^4+4n^3+8n^2+8n}{48(n+1)^2}.$$

Case 2. Let n be odd. Then $\frac{n-3}{2} + 4$ colours are required in an equitable colouring of $W_d(4, n)$. If ✔ = $\{c_1, c_2, c_3, \ldots, c_m, c_{m+1}, c_{m+2}, c_{m+3}, c_{m+4}\}$ be an equitable colouring of $W_d(4, n)$, where, $m = \frac{n-3}{2}$, then the colour classes of $c_1, c_2, c_3, \ldots, c_m$ are 2-element sets, while the colour classes of , c_{m+1}, c_{m+2}, c_{m+3} *and* $c_{m+4}$are singleton sets. Let X be therandom variable which represents the colour of an arbitrarily chosen vertex of $W_d(4, n)$. Then, the corresponding p.m.f of G is,

$$\frac{2}{n+1}, \quad for\ i = 1,2, \ldots\ldots \frac{n-3}{2}$$

$f(i) = p(X = i) = \{ \frac{1}{n+1}, \quad for\ i = \frac{n-3}{2}, \frac{n-3}{2}+1, \frac{n-3}{2}+2, \frac{n-3}{2}+3, \frac{n-3}{2}+4$
$0, \quad elsewhere$

Then the corresponding equitable colouring mean is,

$\mu_{\chi e}(W_d) = (1+2+3+\cdots+\frac{n-3}{2}).\frac{2}{n+1} + (\frac{n-3}{2}+1+\frac{n-3}{2}+2+\frac{n-3}{2}+3+\frac{n-3}{2}+4).\frac{1}{n+1} = \frac{n^2+4n+19}{4(n+1)}$ and $\sigma^2_{\chi e}(W_d) = (1^2+2^2+3^2+\cdots+\frac{(n-3)^2}{4}).\frac{2}{n+1} + ((\frac{n-3}{2}+1)^2+(\frac{n-3}{2}+2)^2+(\frac{n-3}{2}+3)^2+(\frac{n-3}{2}+4)^2).\frac{1}{n+1} - (\frac{n^2+4n+19}{4(n+1)})^2 = \frac{n^4+4n^3+98n^2+188n-675}{48(n+1)^2}$.

2) *Theorem 3.*

For a friendship graph $F_{(4,n)}$, we have

$$\mu_{\chi e}(F_{(4,n)}) = \begin{cases} \frac{2n+3}{n+1}, & if\ n\ is\ even \\ \frac{2n+1}{n+1}, & if\ n\ is\ odd \end{cases}$$

$$\sigma^2_{\chi e}(F_{(4,n)}) = \frac{2n^2+5n}{3(n+1)^2}$$

Proof. Note that the equitable colouring of a friendship graph $F_{(4,n)}$contains 3 colours, say c_1, c_2 and c_3. Then we consider the following cases,

Case 1. If n is even, then $\theta(c_1) = \theta(c_2) = \frac{n}{3}$ and $\theta(c_3) = \frac{n+3}{3}$. Then, the corresponding to the random variable X defined as mentioned above, the p.m.f is given by,

$f(i) = p(X = i) =$

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$$f(i) = p(X = i) = \begin{cases} \frac{n}{3(n+1)}, & for\ i = 1,2 \\ \frac{n+3}{3(n+1)}, & for\ i = 3 \\ 0, & elsewhere \end{cases}$$

Then, the corresponding equitable colouring mean is,

$$\mu_{\chi_e} = (1+2).\frac{n}{3(n+1)} + 3.\frac{n+3}{3(n+1)} = \frac{2n+3}{n+1}$$ and

$$\sigma^2_{\chi_e} = (1^2+2^2).\frac{n}{3(n+1)} + 3^2.\frac{n+3}{3(n+1)} - \left(\frac{2n+3}{n+1}\right)^2 = \frac{2n^2+5n}{3(n+1)^2}.$$

Case 2. If n is odd, then $\theta(c_1) = \frac{n+3}{3}$, $\theta(c_2) = \theta(c_3) = \frac{n}{3}$. Then, corresponding to the random variable X defined as mentioned above, the p.m.f is given by,

$$f(i) = p(X = i) = \begin{cases} \frac{n+3}{3(n+1)}, & for\ i = 1 \\ \frac{n}{3(n+1)}, & for\ i = 2,3 \\ 0, & elsewhere \end{cases}$$

Then, the corresponding equitable colouring mean is,

$$\mu_{\chi_e} = 1.\frac{n+3}{3(n+1)} + (2+3).\frac{n}{3(n+1)} = \frac{2n+1}{n+1}$$ and

$$\sigma^2_{\chi_e} = 1^2.\frac{n+3}{3(n+1)} + (2^2+3^2).\frac{n}{3(n+1)} - \left(\frac{2n+1}{n+1}\right.$$

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$^2 = \frac{2n^2+5n}{}$.

χ_e $3(n+1)$ $3(n+1)$) $n+1$ $3(n+1)^2$

Theorem 4. For the gear graph G_n, we have

$$\mu_{\chi_e}(G_n) = \frac{3n+2}{2(n+1)}$$

$$\sigma^2_{\chi_e}(G_n) = \frac{n^2+2n}{}$$

Proof. Note that the equitable colouring of a gear graph G_n contains 2 colours, say C_1 and C_2. When we subdivide the wheel graph then it becomes a gear graph. Then the corresponding

p.m.f of G is,

$$f(i) = p(X = i) = \begin{cases} \frac{n+2}{2(n+1)}, & for\ i = 1 \\ \frac{n}{2(n+1)}, & for\ i = 2 \\ 0, & elsewhere \end{cases}$$

Hence, the corresponding equitable colouring mean of the gear graph G_n is given by,

$$\mu_{\chi_e} = 1.\frac{n+2}{2(n+1)} + 2.\frac{n}{2(n+1)} = \frac{3n+2}{2(n+1)}$$

and the corresponding equitable colouring variance of G_n is,

$$\sigma^2_{\chi_e} = 1^2.\frac{n+2}{2(n+1)} + 2^2.\frac{n}{2(n+1)} - \left(\frac{3n+2}{2(n+1)}\right)^2 = \frac{n^2+2n}{4(n+1)^2}.$$

Theorem 5. For the sunlet graph S_n, we have,

$$\mu_{\chi_e}(S_n) = \frac{}{n^2+2n+44n}$$

$$\sigma^2_{\chi_e}(S_n) = \frac{n^4 + 20n^2 - 4848n^2}{}$$

Proof. Let $v_1, v_2, \ldots . . v_n$ be the vertices of the graph S_n. *Every* other colour class, with respect to an equitable colouring of S_n, must be a singleton (or) a 2-element set.

Then $\frac{n-2}{2} + 2$ colours are required in an equitable colouring of S_n.

If ✔ $= \{c_1, c_2, c_3, \ldots\ldots, c_m, c_{m+1}, c_{m+2}\}$ be an equitable colouring of , c_{m+1} *and* , c_{m+2} are singleton sets. Let X be the random variable which represents the colour of an arbitrarily chosen vertex of , S_n. Then, the corresponding p.m.f of G is,

$$f(i) = p(X = i) = \begin{cases} \frac{2}{n}, & for\ i = 1,2, \ldots\ldots \frac{n-2}{2} \\ \frac{1}{n}, & for\ i = \frac{n-2}{2} + 1, \frac{n-2}{2} + 2 \\ 0, & elsewhere \end{cases}$$

Then, the corresponding equitable colouring mean is,

$$\mu_{\chi_e} = \left(1 + 2 + 3 + \cdots + \frac{n-2}{2}\right) . \frac{2}{n} + \left(\frac{n-2}{2} + 1 + \frac{n-2}{2} + 2\right) . \frac{1}{n} = \frac{n^2+2n+4}{4n}$$

and the Corresponding equitable colouring variance of S_n is,

$$\sigma^2_{\chi_e} = \left(1^2 + 2^2 + 3^2 + \cdots + \frac{(n-2)^2}{4}\right) . \frac{2}{n} + \left(\left(\frac{n-2}{2} + 1\right)^2 + \left(\frac{n-2}{2} + 2\right)^2\right) . \frac{1}{n} - \left(\frac{n^2+2n+4}{4n}\right)^2 = \frac{n^4+20n^2-48}{48n^2}.$$

CONCLUSION

In this paper, we had discussed about two important statistical parameters, mean and variance which are related to equitable coloring of five different types of graphs.

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