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Homogeneous Quadratic Equation With Three Unknowns $z^2 = 6x^2 - 2y^2$

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ABSTRACT

In this paper, different sets of non-zero distinct integer solutions to the homogeneous quadratic equation with three unknowns given by $z^2 = 6x^2 - 2y^2$ are obtained. A few interesting relations among the solutions are presented. The formulae for generating sequence of integer solutions to the equation in title based on its given solution are exhibited.

Keywords: Homogeneous quadratic, Quadratic with three unknowns, Integer solutions, generation formula

INTRODUCTION

Ternary quadratic equations are rich in variety [1- 4, 17-20].For an extensive review of sizable literature and various problems, one may refer [5-16]. In this communication, we consider yet another interesting homogeneous ternary quadratic equation $z^2 = 6x^2 - 2y^2$ and obtain infinitely many non-trivial integral solutions. A few interesting relations among the solutions are presented. The formulae for generating sequence of integer solutions to the equation in title based on its given solution are exhibited.

METHOD OF ANALYSIS

The homogeneous quadratic equation with three unknowns to be solved is

$$z^2 = 6x^2 - 2y^2 \tag{1}$$

Different sets of solutions in integers to (1) through various ways are illustrated below:

Way 1:
Introduction of the linear transformations

$$x = 2(u+v),\ y = 2u + 6v,\ z = 4w \tag{2}$$

in (1) leads to

$$u^2 - 3v^2 = w^2 \tag{3}$$

which is satisfied by

$$\left.\begin{aligned} v &= 2rs \\ w &= 3r^2 - s^2 \\ u &= 3r^2 + s^2 \end{aligned}\right\} \tag{4}$$

Substituting the above values of u, v, w in (2), the non-zero distinct integral values of x ,y and z are given by

$$\left.\begin{aligned} x &= x(r,s) = 6r^2 + 4rs + 2s^2 \\ y &= y(r,s) = 6r^2 + 12rs + 2s^2 \\ z &= z(r,s) = 12r^2 - 4s^2 \end{aligned}\right\} \quad (5)$$

Thus (5) represents the non-zero integer solutions to (1).

Way 2:

Write (3) as the system of double equations as shown in Table 1 below:

Table 1: System of double equations

System	1	2	3	4	5	6
$u+w$	$3v^2$	v^2	$3v$	1	3	v
$u-w$	1	3	v	$3v^2$	v^2	$3v$

Solving each of the system of equations in Table 1, the corresponding values of u, v, w are obtained. Substituting the values of u, v, w in (2), the respective values of x, y and z are determined. The integer solutions to (1) obtained through solving each of the above system of equations are exhibited below:

Solutions through System 1:

$$x = 12k^2 + 16k + 6$$
$$y = 12k^2 + 24k + 10$$
$$z = 24k^2 + 24k + 4$$

Solutions through System 2:

$$x = 4k^2 + 8k + 6$$
$$y = 4k^2 + 16k + 10$$
$$z = 8k^2 + 8k - 4$$

Solutions through System 3:

$$x = 6v, y = 10v, z = 4v$$

Solutions through System 4:

$$x = 12k^2 + 16k + 6$$
$$y = 12k^2 + 24k + 10$$
$$z = -24k^2 - 24k - 4$$

Solutions through System 5:

$$x = 4k^2 + 8k + 6$$
$$y = 4k^2 + 16k + 10$$
$$z = -8k^2 + 8k + 4$$

Solutions through System6:

$$x = 6v, y = 10v, z = -4v$$

WAY 3:

Rewrite (3) as

$$w^2 + 3v^2 = u^2 * 1 \quad (6)$$

Let

$$u = a^2 + 3b^2 \tag{7}$$

where a and b are non-zero integers.
Consider 1 as

$$1 = \left(\frac{(1 + i\sqrt{3})(1 - i\sqrt{3})}{4} \right) \tag{8}$$

Using (7), (8) in (6) and applying the method of factorization, define

$$(w + i\sqrt{3}v) = (a + i\sqrt{3}b)^2 \frac{(1 + i\sqrt{3})}{2}$$

from which we have

$$\left.\begin{aligned} w &= \frac{1}{2}(a^2 - 6ab - 3b^2) \\ v &= \frac{1}{9}(a^2 + 2ab - 3b^2) \end{aligned}\right\} \tag{9}$$

Using (7) and (9) in (2), the values of x, y and z are given by

$$\left.\begin{aligned} x &= x(a,b) = 3a^2 + 2ab + 3b^2 \\ y &= y(a,b) = 5a^2 + 6ab - 3b^2 \\ z &= z(a,b) = 2a^2 - 12ab - 6b^2 \end{aligned}\right\} \tag{10}$$

Thus (10) represents the non-zero integer solutions to (1).

Note 1:
The integer 1 on the R.H.S. of (6) may also be written as

$$1 = \frac{(1 + i4\sqrt{3})(1 - i4\sqrt{3})}{49}$$

$$1 = \frac{(3r^2 - s^2 + i2rs\sqrt{3})(3r^2 - s^2 - i2rs\sqrt{3})}{(3r^2 + s^2)^2}$$

Following the procedure as above ,one obtains two more sets of integer solutions to (1).

WAY 4:
Rewrite (3) as

$$u^2 - 3v^2 = w^2 * 1 \tag{11}$$

Let

$$w = a^2 - 3b^2 \tag{12}$$

where a and b are non-zero integers.
Take 1 on the R.H.S. of (11) as

$$1 = (2 + \sqrt{3})(2 - \sqrt{3}) \tag{13}$$

Using (12), (13) in (11) and applying the method of factorization, define

$$(u + \sqrt{3}v) = (a + \sqrt{3}b)^2 (2 + \sqrt{3})$$

from which we have

$$\left.\begin{aligned} u &= (2a^2 + 6ab + 6b^2) \\ v &= (a^2 + 4ab + 3b^2) \end{aligned}\right\} \tag{14}$$

Using (12) and (14) in (2), the values of x, y and z are given by

$$\left.\begin{aligned} x(a,b) &= 6a^2 + 20ab + 18b^2 \\ y(a,b) &= 10a^2 + 36ab + 30b^2 \\ z(a,b) &= 4a^2 - 12b^2 \end{aligned}\right\} \quad (15)$$

Thus (15) represents the non-zero integer solutions to (1).
Note 2:

The integer 1 on the R.H.S. of (11) may also be written as

$$1 = (7 + 4\sqrt{3})(7 - 4\sqrt{3})$$

$$1 = \frac{(3r^2 + s^2 + 2rs\sqrt{3})(3r^2 + s^2 - 2rs\sqrt{3})}{(3r^2 - s^2)^2}$$

Following the procedure as above ,one obtains two more sets of integer solutions to (1).

WAY 5:

Introduction of the linear transformations

$$z = 4(u+v),\ y = 4u - 2v,\ x = 2w \quad (16)$$

in (1) leads to

$$w^2 = v^2 + 2u^2 \quad (17)$$

which is satisfied by

$$\left.\begin{aligned} u &= 2rs \\ v &= 2r^2 - s^2 \\ w &= 2r^2 + s^2 \end{aligned}\right\} \quad (18)$$

Substituting the above values of u, v, w in (16), the non-zero distinct integral values of x ,y and z are given by

$$\left.\begin{aligned} z = z(r,s) &= 8r^2 + 8rs - 4s^2 \\ y = y(r,s) &= -4r^2 + 8rs + 2s^2 \\ x = x(r,s) &= 4r^2 + 2s^2 \end{aligned}\right\} \quad (19)$$

Thus (19) represents the non-zero integer solutions to (1).

Generation of Solutions

Different formulas for generating sequence of integer solutions based on the given solution are presented below:
Let $(x_0, y_{0,}, z_0)$ be any given solution to (1)

Formula: 1

Let $(x_{1,} y_1, z_1)$ given by

$$x_1 = 3x_0,\ y_1 = h - 3y_0,\ z_1 = h - 3z_0 \quad (20)$$

be the 2^{nd} solution to (1). Using (20) in (1) and simplifying, one obtains

$$h = 4y_0 + 2z_0$$

In view of (20), the values of y_1 and z_1 are written in the matrix form as

$$(y_1, z_1)^t = M(y_0, z_0)^t$$

where

$$M = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$$ and t is the transpose

The repetition of the above process leads to the n^{th} solutions y_n, z_n given by

$$(y_n, z_n)^t = M^n (y_0, z_0)^t$$

If α, β are the distinct eigenvalues of M, then

$$\alpha = 3, \beta = -3$$

We know that

$$M^n = \frac{a^n}{(\alpha-\beta)}(M-\beta I) + \frac{\beta^n}{(\beta-\alpha)}(M-\alpha I),\ I = 2\times 2 \text{ Identity matrix}$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = 3^n x_0$$

$$y_n = \left(\frac{2\alpha^n + \beta^n}{3}\right) y_0 + \left(\frac{\alpha^n - \beta^n}{3}\right) z_0$$

$$z_n = 2\left(\frac{\alpha^n - \beta^n}{3}\right) y_0 + \left(\frac{\alpha^n + 2\beta^n}{3}\right) z_0$$

Formula: 2

Let (x_1, y_1, z_1) given by

$$x_1 = h - x_0,\ y_1 = h + y_0,\ z_1 = z_0 \qquad (21)$$

be the 2^{nd} solution to (1). Using (21) in (1) and simplifying, one obtains

$$h = 3x_0 + y_0$$

In view of (13), the values of x_1 and y_1 are written in the matrix form as

$$(x_1, y_1)^t = M(x_0, y_0)^t$$

where

$$M = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \text{ and t is the transpose}$$

The repetition of the above process leads to the n^{th} solutions x_n, y_n given by

$$(x_n, y_n)^t = M^n (x_o, y_0)^t$$

If α, β are the distinct eigenvalues of M, then

$$\alpha = 2+\sqrt{3}, \beta = 2-\sqrt{3}$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = \frac{\alpha^n + \beta^n}{2} x_0 + \frac{\alpha^n - \beta^n}{2\sqrt{3}} y_0$$

$$y_n = \frac{(\alpha^n - \beta^n)\sqrt{3}}{2} x_0 + \frac{\alpha^n + \beta^n}{2} y_0$$

$$z_n = z_o$$

Formula: 3

Let $(x_1, y_1 z_1)$ given by

$$x_1 = -x_0 + h,\ y_1 = y_0,\ \ z_1 = 2h + z_0 \qquad (22)$$

be the 2^{nd} solution to (1). Using (22) in (1) and simplifying, one obtains

$$h = 6x_0 + 2z_0$$

In view of (14), the values of x_1 and z_1 are written in the matrix form as

$$(x_1, z_1)^t = M(x_0, z_0)^t$$

where

$$M=\begin{pmatrix}5 & 2\\ 12 & 5\end{pmatrix}$$ and t is the transpose

The repetition of the above process leads to the n^{th} solutions x_n, z_n given by

$$(x_n, z_n)^t = M^n (x_0, z_0)^t$$

If α, β are the distinct eigen values of M, then

$$\alpha = 5+2\sqrt{6},\ \beta = 5-2\sqrt{6}$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = \left(\frac{\alpha^n+\beta^n}{2}\right)x_0 + \left[\frac{\alpha^n-\beta^n}{2\sqrt{6}}\right]z_0$$

$$y_n = y_0$$

$$z_n = \frac{3}{\sqrt{6}}(\alpha^n-\beta^n)x_0 + \left(\frac{\alpha^n+\beta^n}{2}\right)z_0$$

Formula: 4

Let

$$x_1 = h-3x_0,\ y_1 = 3y_0+h,\ z_1 = h+3z_0 \tag{23}$$

be the second solution of (1). Substituting (23) in (1) and performing a few calculations , it is seen that

$$h = 12x_0 + 4y_0 + 2z_0 \tag{24}$$

Using (24) in (23) ,the second solution (x_1, y_1, z_1) to (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M(x_0, y_0, z_0)^t$$

where t is the transpose and

$$M=\begin{bmatrix}9 & 4 & 2\\ 12 & 7 & 2\\ 12 & 4 & 5\end{bmatrix}$$

The repetition of the above process leads to the general solution (x_n, y_n, z_n) to (1) written in the matrix form as

$$(x_n, y_n, z_n)^t = M^n (x_0, y_0, z_0)^t$$

where

$$M^n = \begin{bmatrix} y_{n-1} & 2x_{n-1} & x_{n-1}\\ 6x_{n-1} & \frac{2y_{n-1}+3^n}{3} & \frac{y_{n-1}-3^n}{3}\\ 6x_{n-1} & \frac{2(y_{n-1}-3^n)}{3} & \frac{y_{n-1}+2\ 3^n}{3}\end{bmatrix}$$

in which

$$y_{n-1} = \frac{(9+2\sqrt{18})^n + (9-2\sqrt{18})^n}{2},$$

$$x_{n-1} = \frac{(9+2\sqrt{18})^n - (9-2\sqrt{18})^n}{2\sqrt{18}},\ n=1,2,3,...$$

CONCLUSION

In this paper, an attempt has been made to obtain non-zero distinct integer solutions to the ternary quadratic diophantine equation $z^2 = 6x^2 - 2y^2$ representing homogeneous cone. As there are varieties of cones, the readers may search for other forms of cones to obtain integer solutions for the corresponding cones.

REFERENCES

[1]. Bert Miller, "Nasty Numbers", The Mathematics Teacher, Vol-73, No.9,p.649, 1980.
[2]. Bhatia .B.L and Supriya Mohanty, "Nasty Numbers and their Characterisation" Mathematical Education, Vol-II, No.1, p.34-37,July-September, 1985.
[3]. Carmichael. R.D., The theory of numbers and Diophantine Analysis, New York, Dover,1959.
[4]. Dickson. L.E., History of Theory of numbers, vol.2:Diophantine Analysis, New York, Dover, 2005.
[5]. Gopalan M.A., Manju somnath, and Vanitha.M., Integral Solutions of $kxy + m(x+y) = z^2$, Acta Ciencia Indica, Vol 33, No. 4,1287-1290, (2007).
[6]. Gopalan M.A., Manju Somanath and V. Sangeetha, On the Ternary Quadratic Equation $5(x^2+y^2)-9xy=19z^2$,IJIRSET, Vol 2, Issue 6,2008-2010,June 2013.
[7]. Gopalan M.A., and A. Vijayashankar, Integral points on the homogeneous cone $z^2 = 2x^2 + 8y^2$, IJIRSET, Vol 2(1), 682-685,Jan 2013.
[8]. Gopalan M.A., S. Vidhyalakshmi, and V. Geetha, Lattice points on the homogeneous cone $z^2 = 10x^2 - 6y^2$, IJESRT, Vol 2(2), 775-779,Feb 2013.
[9]. Gopalan M.A., S. Vidhyalakshmi and E. Premalatha , On the Ternary quadratic Diophantine equation $x^2 + 3y^2 = 7z^2$, Diophantus.J.Math1(1),51-57,2012.
[10]. Gopalan M.A., S. Vidhyalakshmi and A.Kavitha, Integral points on the homogeneous cone $z^2 = 2x^2 - 7y^2$, Diophantus.J.Math1(2),127-136,2012.
[11]. M.A. Gopalan and G. Sangeetha, Observations on $y^2 = 3x^2 - 2z^2$, Antarctica J. Math., 9(4),359-362,(2012).
[12]. Gopalan M.A., Manju Somanath and V. Sangeetha , Observations on the Ternary Quadratic Diophantine Equation $y^2 = 3x^2 + z^2$,Bessel J.Math., 2(2),101-105,(2012).
[13]. Gopalan M.A., S. Vidhyalakshmi and E. Premalatha , On the Ternary quadratic equation $x^2 + xy + y^2 = 12z^2$,Diophantus.J.Math1(2),69-76,2012.
[14]. Gopalan M.A., S. Vidhyalakshmi and E. Premalatha, On the homogeneous quadratic equation with three unknowns $x^2 - xy + y^2 = (k^2+3)z^2$, Bulletin of Mathematics and Statistics Research, Vol 1(1),38-41,2013.
[15]. Meena. K, Gopalan M.A., S. Vidhyalakshmi and N. Thiruniraiselvi, Observations on the quadratic equation $x^2 + 9y^2 = 50z^2$, International Journal of Applied Research , Vol 1(2),51-53,2015.
[16]. R. Anbuselvi and S.A. Shanmugavadivu, On homogeneous Ternary quadratic Diophantine equation $z^2 = 45x^2 + y^2$, IJERA, 7(11), 22-25, Nov 2017.
[17]. Mordell L.J., Diophantine Equations, Academic press, London (1969).
[18]. Nigel, P. Smart, The Algorithmic Resolutions of Diophantine Equations, Cambridge University Press, London 1999.
[19]. Telang, S.G., Number Theory, Tata Mc Graw-hill publishing company, New Delhi, 1996.
[20]. S. Vidhyalakshmi, T. Mahalakshmi, "A Study On The Homogeneous Cone $x^2 + 7y^2 = 23z^2$", International Research Journal of Engineering and Technology (IRJET), Volume 6, Issue 3, Pages 5000-5007, March 2019.