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Note on Dirichlet Averages of the S-function

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ABSTRACT

In this study, we investigate Dirichlet Averages of the S-function developed and analysed by Saxena and Daiya [4], where more general Prabhakar integrals are used in place of Riemann-Liouville integrals. In terms of Mittag-Leffler functions, we examine and talk about its characteristics. Additionally, we demonstrate some uses for these Dirichlet Averages of the S-function in difference-differential equations governing the dynamics of generalised renewal stochastic processes as well as in some classical mathematical physics equations, such as the heat and free electron laser equations.

Key Words: S-function, Dirichlet Averages , Riemann Liouville fractional Integrals, Mittag-Leffler Function.

INTRODUCTION

A function's Dirichlet averages are an integral average of a specific kind in relation to the Dirichlet measure. Carlson first mentioned the Dirichlet average in 1977. Numerous researchers have looked into it, including Carlson [1, 2, 4], Zu Castel [5, 6] Massopust and Forster [6, 7], Neuman and Vanfleet [8], and others. Carlson provides a thorough and in-depth analysis of several sorts of Dirichlet averages in his monography [3].

The present paper's goal is to explore the S-function's Dirichlet averages, S-function developed and analysed by Saxena and Daiya [9], The S- functions are recognised to serve crucial roles in numerous applications of the fractional calculus, much like the Mittag-Leffler type functions do. This is mostly caused by their connections to the Mittag-Leffler functions via the Laplace and Fourier transforms.
Riemann-Liouville integrals and Dirichlet integrals, a multivariate integral and a generalisation of a beta integral, are both used in this paper. Finally, using the fractional integrals in particular, we derive representations for the Dirichlet averages $R_k(\beta, \beta^{|}; \mathrm{x}, \mathrm{y})$of the S-function.

Definitions and preliminaries used in the paper:

S-Function:

The S -function defined and studied by Saxena and Daiya [9] as follows:

$$\underset{p,q}{\overset{(\alpha,\beta,\gamma,\tau)}{S}}\left[\begin{matrix} a_1, a_2, .. a_p \\ b_1, b_2, ... b_q \end{matrix}, \ x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n, (a_2)_n, \dots\dots\dots..(a_p)_n (\gamma)_{n\tau,k} x^n}{(b_1)_n, (b_2)_n, \dots\dots\dots..(b_q)_n \quad \Gamma(n\alpha+\beta) n!} \quad (2.1)$$

Where, $k \in R, \alpha, \beta, \gamma, \tau \in C.\ R(\alpha) > 0, a_1, a_2, .. a_p, b_1, b_2, ... b_q, R(\alpha) > kR(\tau)$ and $p < q + 1$.The Pochhammer symbol $(\tau)_\mu$ defined interms of gamma function as follows:

$$(\boldsymbol{\tau})_{\boldsymbol{\mu}} = \frac{\boldsymbol{\Gamma(\tau+\mu)}}{\boldsymbol{\Gamma(\tau)}} = \begin{cases} \mathbf{1}, & \boldsymbol{\mu = 0, \tau, \mu \in c} \\ \boldsymbol{\tau(\tau+1)(\tau+2)\dots\dots(\tau+\mu-1)} \end{cases}$$

Standard simplex in R^n , n ≥ 1: We denote the standard simplex in R^n $, n \geq 1$ by E = E_n = $(u_1 , u_2 , \dots u_n)$; $u_1 \geq 0 , u_2 \geq 0 ,\dots ,u_n \geq 0$ and $u_1 + u_2 + u_3 + u_n \leq 1\}$.

Dirichlet Measures : let b ϵc^k> ; K ≥ 2 and let E = E_{k-1} be the standard simplex in R^{k-1} $.The$ complex measure μ_b defined by [1]

$$d_{\mu_b}(\mathrm{u}) = \frac{1}{B(b)} u_1^{\ b_1-1} u_2^{\ b_2-1} u_3^{\ b_3-1} \dots u_k^{\ b_{k-1}-1} (1-u_1 , 1-u_2 , \dots 1-u_{k-1})^{b_k-1} d_{u_1} d_{u_2} d_{u_3} \dots d_{u_{k-1}}.$$

Here B(b) = B(b_1 , b , ... b_k) = $\frac{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_k)}{\Gamma(b_1+b_2+\cdots.b_k)}$

C> = {zєc : z≠0 }

Dirichlet average: let Ώ be a convex set in C and let z= (z_1 , z_2 , ... z_n) єΏ^n , n≥2, and let f be a measurable function on Ώ .Define

F(b; z) = $\int_{E_{n-1}} f(uoz)\, d_{\mu_b}$(u).where d_{μ_b}(u) is a Dirichlet Measure.

B(b) = B(b_1 , b , ... b_n) = $\frac{\Gamma(b_1)\Gamma(b_2)\ldots\Gamma(b_n)}{\Gamma(b_1+b_2+\cdots b_n)}$, R($b_j$) > 0, j = 1,2,3...,n

And $uoz = \sum_{j=1}^{n-1} u_j z_j + (1 - u_1 \ \ldots - u_{n-1})\, z_n$.

For n= 1, $f(b;z)$ = f(z) , for n = 2 , we have

$d_{\mu_{\beta,\beta'}}$(u) = $\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} u^{\beta-1}(1-u)^{\beta'-1}$ d(u).

Carlson [3] investigated the average for

$f(z) = z^k$, kєR,

R_k(b;z) = $\int_{E_{n-1}} (uoz)^k\, d_{\mu_b}$(u) , (kєR)

and for n= 2 , Carlson proved that

$R_k(\beta,\beta'$;x,y) = $\frac{1}{B(\beta,\beta')}\int_0^1 [ux+(1-u)y]^k u^{\beta-1}(1-u)^{\beta'-1}$ d(u),

Where β,β'є C , min [R(β),R(β')] > 0 , $x, y \epsilon R$.

MAIN RESULTS

In this section, we are devoted to the study of the Dirichlet averages of the s- function (2.1) in the form

$$M_{p,q}^{(\alpha,\beta,\gamma,\tau)}\left[\begin{matrix}a_1, a_2, .. a_p\\ b_1, b_2, ... b_q\end{matrix}\Big|\beta,\beta'; \mathrm{x},\mathrm{y}\right] = \int_{E_1} \mathrm{p\psi q}(uoz)\, d_{\mu_{\beta\beta'}}(\mathrm{u}) \qquad (3.1)$$

Where R(β) > 0, R(β') > 0 ; $x, y \epsilon R$ and β,β'є C.

Reimann-Liouville fractional integral of order αє C, R(α) > 0 [10].

$$(I_{a+}^{\alpha} f)x = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\, dt ,\ (x > a , a \epsilon R) \qquad (3.2)$$

Representation of R_k and $M_{p,q}^{(\alpha,\beta,\gamma,\tau)}$ in terms of Reimann-Liouville fractional integrals. In this section we deduced representations for the Dirichlet averages $R_k(\beta,\beta',x,y)$ and $M_{p,q}^{(\alpha,\beta,\gamma,\tau)}(\beta,\beta';\mathrm{x},\mathrm{y})$ with fractional integral operators.

Theorem 1: Let β,β' є Complex numbers , R(β) > 0, R(β') > 0 , and x,y be real numbers such that $x > y$ and 1+$\sum_{j=1}^{q} B_j$ - $\sum_{j=1}^{p} A_j$≥ 0,and $M_{p,q}^{(\alpha,\beta,\gamma,\tau)}(\beta,\beta';\mathrm{x},\mathrm{y})$ and I_{a+}^{α} be given by (3.1) and (3.2) respectively . Then the Dirichlet average of the generalized Fox- wright functions is given by

$$M_{p,q}^{(\alpha,\beta,\gamma,\tau)}\left[\begin{matrix}a_1, a_2, .. a_p\\ b_1, b_2, ... b_q\end{matrix}\Big|\beta,\beta'; \mathrm{x},\mathrm{y}\right] = \int_{E_1} \mathrm{p\psi q}(uoz)\, d_{\mu_{\beta\beta'}}(\mathrm{u})$$

$$= \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\left[\left(I_{0+}^{\alpha}\, \mathrm{p\psi q}\left(\begin{matrix}a_1, a_2, .. a_p\\ b_1, b_2, ... b_q\end{matrix}\Big| z\right)\right)\right]$$

Where β,β'є C , R(β) > 0, R(β') > 0 , x,y є R and 1+$\sum_{j=1}^{q} B_j$ - $\sum_{j=1}^{q} A_j$≥ 0 (equality only holds for appropriately bounded z).

Proof :According to equation (3.1) and3.2) we have,

$$M_{p,q}^{(\alpha,\beta,\gamma,\tau)}\left[\begin{matrix}a_1, a_2, .. a_p\\ b_1, b_2, ... b_q\end{matrix}\Big|\beta,\beta'; \mathrm{x},\mathrm{y}\right] =$$

$$\frac{1}{B(\beta,\beta')}\sum_{n=0}^{\infty} \frac{(a_1)_n ,(a_2)_n ,..........(a_p)_n}{(b_1)_n ,(b_2)_n ,..........(b_q)_n} \frac{(\gamma)_{n\tau,k} x^n}{\Gamma(n\alpha+\beta)n!}\int_0^1 [y+u(x-y)]^n u^{\beta-1}(1-u)^{\beta'-1}\, \mathrm{d(u)}.$$

$$\overset{(\alpha,\beta,\gamma,\tau)}{\underset{p,q}{M}}\left[\begin{matrix}a_1, a_2, .. a_p\\ b_1, b_2, ... b_q\end{matrix}|\boldsymbol{\beta},\boldsymbol{\beta}^{|};\mathbf{x},\mathbf{y})\right] =$$

$$= \frac{\boldsymbol{\Gamma}(\boldsymbol{\beta}+\boldsymbol{\beta}^{|})}{\boldsymbol{\Gamma}(\boldsymbol{\beta}^{|})\boldsymbol{\Gamma}(\boldsymbol{\beta})}\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\ldots\ldots(a_p)_n(\gamma)_{n\tau,k}x^n}{(b_1)_n,(b_2)_n,\ldots\ldots(b_q)_n\quad\Gamma(n\alpha+\beta)n!}$$

$$\int_0^1[\boldsymbol{y}+\boldsymbol{u}(\boldsymbol{x}-\boldsymbol{y})]^n\boldsymbol{u}^{\boldsymbol{\beta}-\mathbf{1}}(\mathbf{1}-\boldsymbol{u})^{\boldsymbol{\beta}^{|}-\mathbf{1}}\,\mathrm{d(u)}.$$

$\boldsymbol{Put}\ \boldsymbol{u}(\boldsymbol{x}-\boldsymbol{y}) = \boldsymbol{t}$ in above equation , we get

$$\overset{(\alpha,\beta,\gamma,\tau)}{\underset{p,q}{M}}\left[\begin{matrix}a_1, a_2, .. a_p\\ b_1, b_2, ... b_q\end{matrix}|\beta,\beta^{|};\mathrm{x,y})\right] = \int_{E_1}\mathrm{p\psi q}(uoz)\,d_{\mu_{\beta\beta^{|}}}(\mathrm{u})$$

$$= \frac{\Gamma(\beta+\beta^{|})}{\Gamma(\beta)(x-y)^{\beta+\beta^{|}-1}}\left[(I_{0+}^{\alpha}\,\mathrm{p\psi q}\left(\begin{matrix}a_1, a_2, .. a_p\\ b_1, b_2, ... b_q\end{matrix}\middle|z\right)\right]$$

This proves the theorem.

CONCLUSION

The Mittag-Leffler function and its generalisation, the S-function, are essential to fractional calculus. It has been shown that one can use these functions to express the solution of a number of fundamental linear differential equations. These functions behave as a generalisation of the exponential function while solving a fractional differential equation. These functions are therefore essential to the fractional calculus. This paper investigates several internal links between the S-function and Hilfer derivatives of the generalised fractional integration connected to the Gauss-hypergeometric function, in order to facilitate future research.

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