International Journal of Enhanced Research in Educational Development (IJERED) ISSN: 2320-8708, Vol. 11 Issue 1, Jan-Feb, 2023, Impact Factor: 7.326

Around a Nearly Norden Manifold

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ABSTRACT

For this, we have studied a manifold that is practically a Norden manifold and obtained several significant theorems and findings. As a further contribution, we have defined a linear connection on a nearly Norden manifold and obtained some significant findings about this connection.

Keywords: Ricmannian, Curvature tensor, Torsion.

INTRODUCTION

We consider n-dimensional real differentiable manifold V_n of class C^∞ endowed with a C^∞ real vector valued function F such that

$$F^2(X) = -X \quad \ldots(1)$$

i.e. $\bar{\bar{X}} = -X$

where $F(X) \underset{=}{def} \bar{X}$ and 'n' is even

On V_n there exists a Ricmannian metric "g" satisfying

$$g(\bar{x}, \bar{y}) = -g(x, y) \quad \ldots(2)$$

then V_n satisfying above two equations is called an almost Norden manifold. Here equation (2) is equivalent to

$$g(\bar{x}, y) = g(x, \bar{y}) \quad \ldots(3)$$

Here {F, g} is called an almost Norden structure on V_n. We can easily verify their structure is not ∀ unique on V_n.

We define another structure {F, g} on V_n such that

$$\mu F \underset{=}{def} F\mu \quad \ldots(4)$$

$$g(x, y) \underset{=}{def} g(\mu x, \mu y) \quad \ldots(5)$$

where x, y are arbitrary vector fields on V_n and μ is a non-singular tensor of type (1, 1).

By (4) we get $(\mu 'F)'F = (F\mu)'F$

$\Rightarrow \quad \mu('F\,'F) = F(\mu\,'F)$

$\Rightarrow \quad \mu\,'F^2 = -(F\mu) = (F\,F)\mu = F^2\mu$

$\Rightarrow \quad \mu\,'F^2 = -\mu$

$\Rightarrow \quad 'F^2 = -I_n \quad \ldots(6)$

By (5) we get $´g(´Fx, ´Fy) = g(\mu ´Fx, \mu ´Fy)$

$= g(F\mu x, F\mu y)$

$= -g(\mu x, \mu y)$

$= -\bar{g}(x, y)$

i.e., $\bar{g}(´Fx, ´Fy) = -\bar{g}(x, y) \quad \ldots(7)$

Equation (6) and (7) shows that $\{´F, \bar{g}\}$ is also an almost Norden structure on V_n.

Hence, almost Norden structure is not unique.

We define associate tensor of (1, 1) type tensor field F as

$'F(x, y) = g(\bar{x}, y)$ then we have

$'F(x, y) = g(\bar{x}, y) = g(x, \bar{y}) = g(\bar{y}, x) = 'F(y, x)$

i.e. $'F(x, y) = 'F(y, x)$

i.e. $'F(x, y)$ is symmetric,

and also we have

$'F(\bar{x}, \bar{y}) = g(\bar{x}, \bar{y}) = g(-x, \bar{y}) = -g(x, \bar{y})$

$= -g(\bar{x}, y)$

$= -'F(x, y)$

i.e. $'F(\bar{x}, \bar{y}) = -'F(x, y)$

$\therefore$ $'F$ is pure in both slots.

Theorem 1 : If D is Ricmannian connection on an almost Norden manifold V_n then we have

(a) $(D_x 'F)(y, z) = g(D_xF)(y, z)$

(b) $(D_x 'F)(\bar{y}, z) = -(D_x 'F)(y, \bar{z})$

(c) $(D_x 'F)(\bar{y}, \bar{z}) = (D_x 'F)(y, z)$

Proof :(a) We have $'F(y, z) = g(\bar{y}, z)$

$(D_x\, 'F)(y, z) + 'F(D_x y, z) + 'F(y, D_x z) = g((D_x F) y, z) + g(\overline{D_X y}, z) + g(\bar{y}, D_x z)$

$\Rightarrow$ $(D_x 'F)(y, z) = g((D_xF) y, z)$

(b) Replacing y by $\bar{y}$ in result (a)

$(D_x 'F)(\bar{y}, z) = g((D_xF)\ \bar{y}, z)$

$= -g(\overline{(D_X F) y}, z)$

$= -g((D_xF) y, \bar{z})$

$= -(D_x 'F)(y, \bar{z})$

(c) Replacing z by $\bar{z}$ in (b) we get the required result.

Theorem 2 : On an almost Norden manifold V_n we have

(a) $N(\bar{x}, \bar{y}) = -\overline{N(x, y)} = -\overline{N(\bar{x}, y)} = -\overline{N(x, \bar{y})}$

(b) $'N(\bar{x}, \bar{y}, z) = -'N(x, y, \bar{z}) = -'N(\bar{x}, y, \bar{z}) = -'N(x, \bar{y}, \bar{z})$

(c) $'N(\bar{x}, \bar{y}, \bar{z}) = 'N(x, y, z) = 'N(\bar{x}, y, z) = 'N(x, \bar{y}, z)$

where $'N(x, y, z) \underset{=\!=}{\text{def}} g(N(x, y), z)$.

Proof : We know that Nijenheu's tensor w.r.t. (1, 1) type tensor field F is defined as

$N(x, y) = [Fx, Fy] + F'[x, y] - F[Fx, y] - F[x, Fy]$

i.e. $N(x, y) = [\bar{x}, \bar{y}] + \overline{[x, y]} - \overline{[\bar{x}, y]} - \overline{[x, \bar{y}]}$

Results (a), (b), (c) can be easily derived from this definition.

Theorem 3 : On an almost Norden manifold if V is constravarient almost analytic term L_Vg is pure in both slots.

Proof : Since V is contra variant almost analytic, therefore we get

$L_VF = 0 \quad \Leftrightarrow \quad (L_VF)x = 0$

$\Leftrightarrow \quad [v, \bar{x}] = \overline{[v, x]}$

where "L" is Lic derivative and x is arbitrary vector field.

Taking Lic derivative of $g(\bar{x}, \bar{y})$ w.r.t. V

$L_V(g(\bar{x}, \bar{y})) = (L_Vg)(\bar{x}, \bar{y}) + g(L_V \bar{x}, \bar{y}) + g(\bar{x}, L_V \bar{y})$

$V(g(\bar{x}, \bar{y})) = (L_Vg)(\bar{x}, \bar{y}) + g([v, \bar{x}], \bar{y}) + g(\bar{x}, [v, \bar{y}])$

$\Rightarrow V(g(\bar{x}, \bar{y})) = (L_Vg)(\bar{x}, \bar{y}) + g(\overline{[v, x]}, \bar{y}) + g(\bar{x}, \overline{[v, y]})$

$\Rightarrow (L_V g)(\bar{x}, \bar{y}) = V(g(\bar{x}, \bar{y})) - g(\overline{[v, x]}, \bar{y}) - g(\bar{x}, \overline{[v, y]})$

$\Rightarrow (L_V g)(\bar{x}, \bar{y}) = -V(g(x, y)) + g([v, x], y) + g(x, [v, y])$

$\Rightarrow (L_V g)(\bar{x}, \bar{y}) = -(L_V g)(x, y).$

Theorem 4 : If the vector field V is both contra-variant almost analytic and covariant almost analytic then $d'F(x, y, v) = z((D_{\bar{x}} v)y - (D_x v)\bar{y})$

where $v(x) = g(x, y)$ and D is Riemannian connection on V_n.

Proof : Since V is contra-variant almost analytic

$\therefore$ we have $L_V F = 0 \quad \Leftrightarrow \quad (L_V F)x = 0 \quad \Leftrightarrow \quad [v, \bar{x}] = \overline{[v, x]}$

$\Leftrightarrow \quad D_{\bar{x}} v = (D_V F)x + \overline{D_x v}$

$g(D_{\bar{x}} v, y) = g((D_V F)x, y) + g(\overline{D_x v}, y)$

$\Rightarrow \quad (D_{\bar{x}} v)y = (D_{V'}F)(x, y) + g(D_x v, \bar{y})$

$\Rightarrow \quad (D_{\bar{x}} v)(y) = (D_V{}'F)(x, y) + g(D_x v)(\bar{y})$

$\Rightarrow \quad (D_V{}'F)(x, y) = (D_{\bar{x}} v) y - (D_x v)(\bar{y})$ …(1)

More we have used the result

$$(D_x v)(y) = g(D_x V, y)$$

Now, it is given "v" is covariant almost analytic,

$\therefore$ we have

$v((D_x F) y - (D_y F) x) = (D_{\bar{x}} v) y - (D_x v)\bar{y}$

$\Rightarrow \quad v((D_x F) y - (D_y F) x) = (D_{\bar{x}} v) y - (D_x v)\bar{y}$

$\Rightarrow \quad (D_x{}'F)(y, v) - (D_y{}'F)(x, v) = (D_{\bar{x}} v)(y) - (D_x v)(\bar{y})$

$\Rightarrow \quad (D_x{}'F)(y, v) + (D_y{}'F)(x, v) = (D_{\bar{x}} v)(y) + (D_x v)(\bar{y})$ …(2)

Adding (1) and (2) we get the required result.

Theorem 5 : If 1-form w is covariant almost analytic on an almost Norden manifold V_n, then

$$'N(x, y, w) = 0$$

where $\quad w(x) \underset{=}{\text{def}} g(x, w)$

$$'N(x, y, w) = g(N(x, y), w)$$

Proof : We have

$N(x, y) = (D_{\bar{x}} F) y - (D_{\bar{y}} F) x + \overline{(D_y F)x} - \overline{(D_x F)y}$

$\Rightarrow g(N(x, y), w) = g((D_{\bar{x}} F) y, w) - g((D_{\bar{y}} F) x, w) + g(\overline{(D_y F)x}, w) - g(\overline{(D_x F)y}, w)$

$\Rightarrow \ 'N(x, y, w) = w((D_{\bar{x}} F) y) - w((D_{\bar{y}} F) x) - w((D_y F)\bar{x}) + w((D_y F)\bar{y})$ …(1)

Since "w" is covariant almost analytic

$\therefore$ we have

$w(D_x F) y - w(D_y F) x = (D_{\bar{x}} w) y - (D_x w)\bar{y}$ …(2)

Replacing x by $\bar{x}$ in (2)

$w((D_{\bar{x}} F) y) - w((D_y F)\bar{x}) = -(D_x w) y - (D_{\bar{x}} w)\bar{y}$ …(3)

Replacing y by $\bar{y}$ in (2)

$w((D_x F)\bar{y}) - w((D_{\bar{y}} F) x) = (D_{\bar{x}} w)\bar{y} + (D_x w) y$ …(4)

equation (3) + (4) we get

$((D_{\bar{x}} F) y) - w((D_y F)\bar{x}) + w((D_x F)\bar{y}) - w((D_{\bar{y}} F) x) = 0$ …(5)

Using equation (5) in (1) we get

$$´N(x, y, w) = 0.$$

Theorem 6 : On an almost Norden manifold we have

$(K_{xy}\ ´F)(z, W) = ´K(x, y, \bar{z}, w) – ´K(x, y, z, \bar{w})$

where$K_{xy} = D_xD_y – D_yD_x – D_{[x, y]}$.

Proof : We have $K_{xy}z = K(x, y, z)$

where$K_{xy} = D_xD_y – D_yD_x – D_{[x, y]}$

$$K_{xy}\bar{z} = (Kxy\ F)(z) + F(K_{xy} z)$$

$\Rightarrow \quad K(x, y, \bar{z}) = (K_{xy} F)(z) + \overline{K(x, y, z)}$

$\Rightarrow \quad g(K(x, y, \bar{z}), w) = g((K_{xy} F)(z), w) + g(\overline{K(x, y, z)}, w)$

$\Rightarrow \quad ´K(x, y, \bar{z}, w) = (K_{xy}\ ´F)(z, w) + K(x, y, z, \bar{w})$

$\Rightarrow \quad (K_{xy}\ ´F)(z, w) = ´K(x, y, \bar{z}, w) – ´K(x, y, z, \bar{w})$

Theorem 7 : If D is Ricmannian connection on an almost Norden manifold V_n we define connection B on V_n as

$$B_xY = D_xy + H(x, y)$$

where "H" is a vector valued bilinear function then its torsion is given by

(i) $S(x, y) = H(x, y) – H(y, x)$

(ii) $´S(x, y, z) = ´H(x, y, z) – ´H(y, x, z)$

Proof : Torsion 'S' of connection B is given by

$S(x, y) = B_xy – B_y x – [x, y]$

$= D_xy + H(x, y) – D_yx – H(y, x) – [x, y]$

$= H(x, y) – H(y, x) + D_x y – D_y x – [x, y]$

$= H(x, y) – H(y, x) + 0$

$S(x, y) = H(x, y) – H(y, x)$

$\Rightarrow \quad g(S(x, y), z) = g(H(x, y), z) – g(H(y, x), z)$

$\Rightarrow \quad ´S(x, y, z) = ´H(x, y, z) – H(y, x, z).$

Theorem 8 : On an almost Nordem manifold V_n. If the connection B satisfy $B_x g = 0$ then

(a) $´H(x, y, z) + ´H(x, z, y) = 0$

(b) $´S(x, y, z) – ´S(y, z, x) + ´S(z, x, y) = 2\ ´H(x, y, z) = 2g(B_xy, z) – 2g(D_x y, z)$

Proof : Taking covariant derivative of g(y, z) w.r.t. connection B and Riemannion connection D along x we get

$x(g(y, z)) = (B_x g)(y, z) + g(B_x y, z) + g(y, B_x z)$ …(1)

and

$x(g(y, z)) = (D_x g)(y, z) + g(D_x y, z) + g(y, D_x z)$ …(2)

equation (1) – (2)

$0 = g(B_xy – D_x y, z) + g(y, B_x z – D_x z)$

$\Rightarrow \quad 0 = g(H(x, y), z) + g(y, H(x, z))$

$\Rightarrow \quad 0 = ´H(x, y, z) + ´H(x, z, y)$ …(3)

Now we have

$S(x, y) = H(x, y) – H(y, x)$

$´S(x, y, z) = ´H(x, y, z) – ´H(y, x, z)$ …(4)

By cyclic rotation of x, y, z in above we get

$´S(y, z, x) = ´H(y, z, x) – ´H(z, y, x)$ …(5)

$´S(z, x, y) = ´H(y, z, x) – ´H(x, z, y)$ …(6)

equation (4) – (5) + (6) we get

$´S(x, y, z) – ´S(y, z, x) + ´S(z, x, y) = 2´H(x, y, z) = 2g(H(x, y), z)$

$= 2g(B_xy – D_xy, z)$

$= 2g(B_xy, z) – 2g(D_xy, z)$

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