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On Nonstandard Type of Integral Representation of Bessel Function

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Abstract: In this paper we introduce some special type of integral representation of Bessel function of first kind and give some nonstandard results applying on parameters p, n and variables x, t ,θ for different nonstandard values (infinitesimals, infinitely close, unlimited,…).

Keywords: Bessel function, generating function, integral representation, nonstandard analysis, infinity closed, infinitesimal, monad, galaxy, unlimited.

1. Introduction

Consider the Bessel equation of the form

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \qquad \dots (1.1.1)$$

The general solution of (1.1.1) is $y = AJ_p(x) + BJ_{-p}(x)$, where A and B are arbitrary constants, and

$$J_p(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(p+r+1)} \left(\frac{x}{2}\right)^{p+2r} \quad , \quad \text{and} \quad J_{-p}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-p+r+1)} \left(\frac{x}{2}\right)^{-p+2r}$$

Where p is nonnegative constant.

Many facts about integral representation of Bessel function can be proved by using its generating function

$$e^{\frac{x}{2}(z-z^{-1})} = \sum_{p=-\infty}^{\infty} z^p J_p \qquad \dots (1.1.2)$$

One of such integral form is given by

$$J_p(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin\theta - p\theta)\, d\theta \qquad \dots (1.1.3)$$

Since Equation (1.1.2) is the general form of Laurent expansion where $c_n(x) = J_p(x)$ and $f(z) = e^{\frac{x}{2}(z-z^{-1})}$, we can write the general integral representation of Bessel function

$$J_p(x) = \frac{1}{2\pi i}\int_C \frac{e^{\frac{x}{2}(z-z^{-1})}}{z^{p+1}}\, dz, \qquad \text{where C is a simple closed curve.}$$

Let $z = e^{i\theta}$, and $\theta \in [0,2\pi]$, then

$$J_p(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta - ip\theta}\, d\theta = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - p\theta)\, d\theta \qquad \dots (1.1.4)$$

Where p is any real value. Hence for $p = 0$ we obtain

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\, d\theta \qquad \dots (1.1.5)$$

Throughout this paper the following definitions and notation of nonstandard analysis will be needed

Every set or elements defined in a classical mathematics are standard.

A real number x is called unlimited if $|x| > r$ for all $r > 0$, by Ω we mean the set of unlimited real numbers.

A real number x is called infinitesimal if $|x| < r$ for all positive standard real number r, by $\odot$ we mean the set of infinitesimals.

Two real numbers x and y are said to be infinitely near if $x - y$ is infinitesimal and is denoted by $x \simeq y$.

If x is a limited real number, then the set of all numbers which are infinitely near to x, is called the monad of x and denoted by mon(x).

If x is a real number x, then the set of all numbers y such that $x - y$ is limited is called the galaxy of x, and denoted by gal(x).

2. Main Results

Lemma 2.1: Let I=[a,b], then the integral of the Bessel function on I for p=0 is given by

$$\int_a^b J_0(x)\,dx = \frac{4}{\pi}\int_0^1 \frac{\sin\left(\frac{b-a}{2}t\right)\cos\left(\frac{b+a}{2}t\right)}{t\sqrt{1-t^2}}dt$$

Proof:

Since by integrating Equation (1.1.3) on I=[a,b] and p=0, we get

$$\int_a^b J_0(x)\,dx = \frac{1}{\pi}\int_0^\pi\int_a^b \cos(x\sin\theta)\,dxd\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin(b\sin\theta) - \sin(a\sin\theta)}{\sin\theta}d\theta \quad \ldots (2.1.1)$$

Put $t = \sin\theta$ then Equation (2.1.1) becomes:

$$\int_a^b J_0(x)\,dx = \frac{2}{\pi}\int_0^1 \frac{\sin(bt) - \sin(at)}{t\sqrt{1-t^2}}dt \quad \ldots (2.2.2)$$

Thus by using trigonometric law: $\sin b - \sin a = 2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{b+a}{2}\right)$

We conclude the required result, hence

$$\int_a^b J_0(x)\,dx = \frac{4}{\pi}\int_0^1 \frac{\sin\left(\frac{b-a}{2}t\right)\cos\left(\frac{b+a}{2}t\right)}{t\sqrt{1-t^2}}dt$$

In the following, by using the previous lemma, some nonstandard results are proved for different nonstandard values of p, θ and x.

Lemma 2.2: Let $p \in \text{mon}(0)$.Then

$$\int_a^b J_\delta(x) \simeq \int_a^b J_0(x)$$

Proof:

Let $p \in \text{mon}(0)$, thenp $\simeq \delta \simeq 0$,$\cos p\theta = \cos\delta\theta \simeq 1$ and $\sin p\theta = \sin\delta\theta \simeq 0$.Therefore

$$\cos(x\sin\theta - p\theta) = \cos(x\sin\theta - \delta\theta) \simeq \cos(x\sin\theta)$$

Thus

$$J_\delta(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - \delta\theta)\,d\theta \simeq \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\,d\theta = J_0(x)$$

Then the integration is well defined and

$$\int_a^b J_\delta(x) \simeq \int_a^b J_0(x)$$

Theorem 2.3: For $\theta = \pi - \delta$ where $\delta \simeq 0$ then

$$J_p(x) \simeq \frac{2\sin\left(\frac{x+p}{2}\pi\right)\cos\left(\frac{x-p}{2}\pi\right)}{\pi(x+p)} \quad \ldots (2.3.1)$$

Moreover

$$J_p(x) \simeq \begin{cases} \cos p\pi & x \in \text{mon}(-p) \\ \dfrac{\sin p\pi}{p\pi} & x \in \text{mon}(p) \\ 1 & x, p \in \text{mon}(0) \end{cases}$$

Proof:

For $\theta = \pi - \delta$ where $\delta \simeq 0$ we have

$\sin(\pi - \delta) \simeq \sin\delta \simeq \delta$and $d\theta = -d\delta$, then by using Equation (1.1.3) we get

$$J_p(x) \simeq \frac{1}{\pi}\int_0^{\pi} \cos\big(x\sin\delta - p(\pi-\delta)\big)\,d\delta \simeq \frac{1}{\pi}\int_0^{\pi}\cos\big((x+p)\delta - p\pi\big)\,d\delta$$

$$\simeq \left[\frac{\sin\big((x+p)\delta - p\pi\big)}{\pi(x+p)}\right]_0^{\pi} = \frac{\sin x\pi + \sin p\pi}{\pi(x+p)} = \frac{2\sin\left(\frac{x+p}{2}\pi\right)\cos\left(\frac{x-p}{2}\pi\right)}{\pi(x+p)}$$

Thus

$$J_p(x) \simeq \frac{2\sin\left(\frac{x+p}{2}\pi\right)\cos\left(\frac{x-p}{2}\pi\right)}{\pi(x+p)}$$

Now,

i- For $x \in \text{mon}(-p)$ let $= \frac{x+p}{2}\pi$, then $y \simeq 0$ and $x = \frac{2}{\pi}y - p$. Therefore

$$J_p(x) \simeq \frac{\sin y \cos\left(\dfrac{\frac{2}{\pi}y - 2p}{2}\pi\right)}{y} \simeq \cos p\pi$$

ii-For $x \in \text{mon}(p)$ we have $\frac{x+p}{2}\pi \simeq \frac{2x}{2}\pi \simeq \frac{2p}{2}\pi$, and $\frac{x-p}{2}\pi \simeq 0$, then $\cos\left(\frac{x-p}{2}\pi\right) \simeq 1$. Therefore

$$J_p(x) \simeq \frac{\sin p\pi}{p\pi}$$

iii- For $x, p \in \text{mon}(0)$ and from case (ii) we get that

$$J_p(x) \simeq \frac{\sin x\pi}{x\pi} \simeq 1$$

Theorem 2.4: Let I=[a,b], and $\theta \simeq 0$ then the integral of $J_p(x)$ on I is given as follows

$$\int_a^b J_p(x)\,dx \simeq \int_a^b J_0(x)\,dx + p\left(\frac{\sin a\pi}{a\pi} - \frac{\sin b\pi}{b\pi}\right)$$

Proof:

Let $\theta \simeq 0$then $\cos p\delta \simeq 1$ and $\sin p\delta \simeq p\delta$, therefore by using Equation (1.1.3) we get

$$J_p(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\delta - p\delta)\,d\delta \simeq \frac{1}{\pi}\int_0^{\pi}(\cos(x\sin\delta) + \delta p \sin x\delta)d\delta$$

$$J_p(x) \simeq \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\delta)\,d\delta + \frac{p}{\pi}\int_0^{\pi}\delta\sin x\delta\,d\delta = J_0(x) + \frac{p}{\pi}\int_0^{\pi}\delta\sin x\delta\,d\delta$$

Thus for $x \in I$, we get

$$\int_a^b J_p(x)\,dx \simeq \int_a^b J_0(x)\,dx + \frac{p}{\pi}\int_0^{\pi}\int_a^b \delta\sin x\delta\,dxd\delta \simeq \int_a^b J_0(x)\,dx + \frac{p}{\pi}\int_0^{\pi}[-\cos x\delta]_a^b d\delta$$

$$\simeq \int_a^b J_0(x)\,dx + \frac{p}{\pi}\int_0^{\pi}(\cos a\delta - \cos b\delta)d\delta \simeq \int_a^b J_0(x)\,dx + \frac{p}{\pi}\left[\frac{\sin a\delta}{a} - \frac{\sin b\delta}{b}\right]_0^{\pi}$$

$$\simeq \int_a^b J_0(x)\,dx + p\left(\frac{\sin a\pi}{a\pi} - \frac{\sin b\pi}{b\pi}\right)$$

$$\int_a^b J_p(x)\,dx \simeq \int_a^b J_0(x)\,dx + p\left(\frac{\sin a\pi}{a\pi} - \frac{\sin b\pi}{b\pi}\right)$$

Lemma 2.5: Let I=[0,1], then

$$J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos xt}{\sqrt{1-t^2}}\,dt$$

Proof:

Let $t = \sin\theta$ .Then

$$\frac{2}{\pi}\int_0^1 \frac{\cos xt}{\sqrt{1-t^2}}\,dt = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos(x\sin\theta)}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos(x\sin\theta)}{\cos\theta}\cos\theta\,d\theta$$
$$= \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\,d\theta = J_0(x)$$

Through the following theorems we give an infinitesimal analytical continuation of Bessel function, for any real value p.

Theorem 2.6: If $x \simeq b$ such that $a \le x \le b+\delta$ where $\delta \simeq 0$, then

$$\int_a^{b+\delta} J_p(x)\,dx \simeq \int_a^b J_0(x)\,dx + \delta J_0(b) + p\left(\frac{\sin a\pi}{a\pi} - \frac{\sin b\pi}{(b+\delta)\pi}\right) - \frac{p\delta\cos b\pi}{b+\delta}$$

Proof:

Since $x \simeq b$ such that $a \le x \le b+\delta$ where $\delta \simeq 0$ then by using Theorem (2.5) we get

$$\int_a^{b+\delta} J_p(x)\,dx \simeq \int_a^{b+\delta} J_0(x)\,dx + p\left(\frac{\sin a\pi}{a\pi} - \frac{\sin(b+\delta)\pi}{(b+\delta)\pi}\right)$$

Now by using Equation (2.1.2) we obtain

$$\int_a^{b+\delta} J_p(x)\,dx \simeq \frac{2}{\pi}\int_0^1 \frac{\sin\big((b+\delta)t\big) - \sin(at)}{t\sqrt{1-t^2}}\,dt + p\left(\frac{\sin a\pi}{a\pi} - \frac{\sin(b+\delta)\pi}{(b+\delta)\pi}\right)$$

$$\simeq \frac{2}{\pi}\int_0^1 \frac{\sin bt\cos\delta t + \sin\delta t\cos bt - \sin(at)}{t\sqrt{1-t^2}}\,dt + \frac{p\sin a\pi}{a\pi} - \frac{p}{(b+\delta)\pi}(\sin b\pi\cos\delta\pi + \sin\delta\pi\cos b\pi)$$

For$\delta \simeq 0$ we have $\cos\delta t \simeq 1$, $\sin\delta t \simeq \delta t$, $\cos\delta\pi \simeq 1$ and $\sin\delta\pi \simeq \delta\pi$, therefore

$$\int_a^{b+\delta} J_p(x)\,dx \simeq \frac{2}{\pi}\int_0^1 \frac{\sin bt - \sin(at)}{t\sqrt{1-t^2}}\,dt + \frac{2\delta}{\pi}\int_0^1 \frac{\cos bt}{\sqrt{1-t^2}}\,dt + \frac{p\sin a\pi}{a\pi}$$
$$-\frac{p\sin b\pi}{(b+\delta)\pi} - \frac{p\delta\cos b\pi}{(b+\delta)}$$

Now using Lemma (2.5) and Equation (2.1.2) we get

$$\int_a^{b+\delta} J_p(x)\,dx \simeq \int_a^b J_0(x)\,dx + \delta J_0(b) + p\left(\frac{\sin a\pi}{a\pi} - \frac{\sin b\pi}{(b+\delta)\pi}\right) - \frac{p\delta\cos b\pi}{b+\delta}$$

Theorem 2.7: If $x \simeq a$ such that $a-\delta \le x \le b$ where $\delta \simeq 0$ then

$$\int_{a-\delta}^b J_p(x)\,dx \simeq \int_a^b J_0(x)\,dx + \delta J_0(a) + p\left(\frac{\sin b\pi}{b\pi} - \frac{\sin a\pi}{(a-\delta)\pi}\right) - \frac{p\delta\cos a\pi}{a-\delta}$$

Proof:

Since $x \simeq a$ such that $a-\delta \le x \le b$ where $\delta \simeq 0$ then by using Theorem (2.5) we get

$$\int_{a-\delta}^{b} J_p(x)\,dx \simeq \int_{a-\delta}^{b} J_0(x)\,dx + p\left(\frac{\sin(a-\delta)\pi}{(a-\delta)\pi} - \frac{\sin b\pi}{b\pi}\right)$$

Now by using Equation (2.1.2) we get

$$\int_{a-\delta}^{b} J_p(x)\,dx \simeq \frac{2}{\pi}\int_0^1 \frac{\sin(bt)-\sin((a-\delta)t)}{t\sqrt{1-t^2}}dt + p\left(\frac{\sin(a-\delta)\pi}{(a-\delta)\pi} - \frac{\sin b\pi}{b\pi}\right)$$
$$\simeq \frac{2}{\pi}\int_0^1 \frac{\sin bt - \sin at\cos\delta t + \cos at\sin\delta t}{t\sqrt{1-t^2}}dt + p\left(\frac{\sin a\pi\cos\delta\pi - \cos a\pi\sin\delta\pi}{(a-\delta)\pi} - \frac{\sin b\pi}{b\pi}\right)$$

Since $\delta \simeq 0$, then $\cos\delta t \simeq 1$, $\sin\delta t \simeq \delta t$, $\cos\delta\pi \simeq 1$ and $\sin\delta\pi \simeq \delta\pi$.

Hence

$$\int_{a-\delta}^{b} J_p(x)\,dx \simeq \frac{2}{\pi}\int_0^1 \frac{\sin bt - \sin at}{t\sqrt{1-t^2}}dt + \frac{2\delta}{\pi}\int_0^1 \frac{\cos at}{\sqrt{1-t^2}}dt + \frac{p\sin b\pi}{b\pi} - \frac{p\sin a\pi}{(a-\delta)\pi} - \frac{p\delta\cos a\pi}{(a-\delta)}$$

Now using Lemma (2.5) and Equation (2.1.2) we get

$$\int_{a-\delta}^{b} J_p(x)\,dx \simeq \int_{a}^{b} J_0(x)\,dx + \delta J_0(a) + p\left(\frac{\sin b\pi}{b\pi} - \frac{\sin a\pi}{(a-\delta)\pi}\right) - \frac{p\delta\cos a\pi}{a-\delta}$$

Proposition 2.8: Let x be a real variable. Then

$$\frac{\sin x}{x} - \cos x = \sqrt{\frac{\pi x}{2}}J_{\frac{3}{2}}(x)$$

Proof:

Since $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$, $J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$,

and $J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$

Thus, if $n = \frac{1}{2}$ we have $J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x) - J_{\frac{-1}{2}}(x) = \sqrt{\frac{\pi x}{2}}\left(\frac{\sin x}{x} - \cos x\right)$

Hence

$$\frac{\sin x}{x} - \cos x = \sqrt{\frac{\pi x}{2}}J_{\frac{3}{2}}(x)$$

Theorem 2.9: For $a-\delta \le x \le a$ where $\delta \simeq 0$, then

$$\int_{a-\delta}^{a} J_p(x)dx \simeq \delta J_0(a) + \frac{p\delta\pi}{a-\delta}\sqrt{\frac{a}{2}}J_{\frac{3}{2}}(a\pi)$$

Proof:

Since $a-\delta \le x \le a$ where $\delta \simeq 0$ then by using Theorem (2.5) we get

$$\int_{a-\delta}^{a} J_p(x)\,dx \simeq \int_{a-\delta}^{a} J_0(x)\,dx + p\left(\frac{\sin(a-\delta)\pi}{(a-\delta)\pi} - \frac{\sin a\pi}{a\pi}\right)$$

Now, using Equation (2.1.2) we obtain

$$\int_{a-\delta}^{a} J_p(x)\,dx \simeq \frac{2}{\pi}\int_0^1 \frac{\sin(at)-\sin((a-\delta)t)}{t\sqrt{1-t^2}}dt + p\left(\frac{\sin(a-\delta)\pi}{(a-\delta)\pi} - \frac{\sin a\pi}{a\pi}\right)$$
$$\simeq \frac{2}{\pi}\int_0^1 \frac{\sin at - \sin at\cos\delta t + \cos at\sin\delta t}{t\sqrt{1-t^2}}dt + p\left(\frac{\sin a\pi\cos\delta\pi - \cos a\pi\sin\delta\pi}{(a-\delta)\pi} - \frac{\sin a\pi}{a\pi}\right)$$

Thus, for $\delta \simeq 0$, we have

$$\int_{a-\delta}^{a} J_p(x)\,dx \simeq \frac{2\delta}{\pi}\int_0^1 \frac{\cos at}{\sqrt{1-t^2}}dt + \frac{p\sin a\pi}{(a-\delta)\pi} - \frac{p\delta\cos a\pi}{a-\delta} - \frac{p}{a\pi}\sin a\pi$$

$$\simeq \delta J_0(a) + \frac{p\delta\sin a\pi}{\pi a(a-\delta)} - \frac{p\delta\cos a\pi}{a-\delta} \simeq \delta J_0(a) + \frac{p\delta}{a-\delta}\left(\frac{\sin a\pi}{a\pi} - \cos a\pi\right)$$

Using Proposition (2.8) we get

$$\int_{a-\delta}^{a} J_p\,(x)dx \simeq \delta J_0(a) + \frac{p\delta\pi}{a-\delta}\sqrt{\frac{a}{2}}J_{\frac{3}{2}}(a\pi)$$

Theorem 2.10: For $b \leq x \leq b+\delta$ where $\delta \simeq 0$, then

$$\int_{b}^{b+\delta} J_p\,(x)dx \simeq \delta J_0(b) + \frac{p\delta\pi}{b+\delta}\sqrt{\frac{b}{2}}J_{\frac{3}{2}}(b\pi)$$

Proof:

Since $b \leq x \leq b+\delta$ where $\delta \simeq 0$ then by using Theorem (2.5) we get

$$\int_{b}^{b+\delta} J_p(x)\,dx \simeq \int_{b}^{b+\delta} J_0(x)\,dx + p\left(\frac{\sin b\pi}{b\pi} - \frac{\sin(b+\delta)\pi}{(b+\delta)\pi}\right)$$

Now, using Equation (2.1.2) we get

$$\int_{b}^{b+\delta} J_p(x)\,dx \simeq \frac{2}{\pi}\int_0^1 \frac{\sin(b+\delta)t - \sin bt}{t\sqrt{1-t^2}}dt + p\left(\frac{\sin b\pi}{b\pi} - \frac{\sin(b+\delta)\pi}{(b+\delta)\pi}\right)$$

$$\simeq \frac{2}{\pi}\int_0^1 \frac{\sin bt\cos\delta t + \cos bt\sin\delta t - \sin bt}{t\sqrt{1-t^2}}dt + p\left(\frac{\sin b\pi}{b\pi} - \frac{\sin b\pi\cos\delta\pi + \cos b\pi\sin\delta\pi}{(b+\delta)\pi}\right)$$

Thus for $\delta \simeq 0$ we have

$$\int_{b}^{b+\delta} J_p(x)\,dx \simeq \frac{2\delta}{\pi}\int_0^1 \frac{\cos bt}{\sqrt{1-t^2}}dt + \frac{p\sin b\pi}{b\pi} - \frac{p\delta\sin b\pi}{(b+\delta)\pi} - \frac{p\delta}{b+\delta}\cos b\pi$$

$$\simeq \delta J_0(b) + \frac{p\delta\sin b\pi}{\pi b(b+\delta)} - \frac{p\delta\cos b\pi}{b+\delta} \simeq \delta J_0(a) + \frac{p\delta}{b+\delta}\left(\frac{\sin b\pi}{b\pi} - \cos b\pi\right)$$

Therefore by using Proposition (2.8) we get

$$\int_{b}^{b+\delta} J_p\,(x)dx \simeq \delta J_0(b) + \frac{p\delta\pi}{b+\delta}\sqrt{\frac{b}{2}}J_{\frac{3}{2}}(b\pi)$$

Theorem 2.11: If $a \simeq x \simeq b$ such that $a-\delta \leq x \leq b+\delta$ where $\delta \simeq 0$, then

$$\int_{a-\delta}^{b+\delta} J_p\,(x)dx \simeq \int_a^b J_0\,(x)dx + \delta\big(J_0(a) + J_0(b)\big) + \frac{p\sin a\pi}{\pi(a-\delta)} - \frac{p\sin b\pi}{\pi(b+\delta)} - \frac{p\delta\cos a\pi}{a-\delta} - \frac{p\delta\cos b\pi}{b+\delta}$$

Proof:

Since $a \simeq x \simeq b$ such that $a-\delta \leq x \leq b+\delta$ where $\delta \simeq 0$ then by using Theorems (2.5), (2.9) and (2.10) respectively we get

$$\int_{a-\delta}^{b+\delta} J_p\,(x)dx \simeq \int_{a-\delta}^{a} J_p\,(x)dx + \int_a^b J_p\,(x)dx + \int_b^{b+\delta} J_p\,(x)dx$$

$$\simeq \delta J_0(a) + \frac{p\delta\pi}{a-\delta}\sqrt{\frac{a}{2}}J_{\frac{3}{2}}(a\pi) + \int_a^b J_0(x)\,dx + p\left(\frac{\sin a\pi}{a\pi} - \frac{\sin b\pi}{b\pi}\right) + \delta J_0(b) + \frac{p\delta\pi}{b+\delta}\sqrt{\frac{b}{2}}J_{\frac{3}{2}}(b\pi)$$

Now using Proposition (2.8) we get

$$\int_{a-\delta}^{b+\delta} J_p(x)dx \simeq \int_a^b J_0(x)\,dx + \delta\big(J_0(a)+J_0(b)\big)$$
$$+p\left(\frac{\sin a\pi}{a\pi}-\frac{\sin b\pi}{b\pi}\right)+\frac{p\delta}{b+\delta}\left(\frac{\sin b\pi}{b\pi}-\cos b\pi\right)+\frac{p\delta}{a-\delta}\left(\frac{\sin a\pi}{a\pi}-\cos a\pi\right)$$

Therefore

$$\int_{a-\delta}^{b+\delta} J_p(x)dx \simeq \int_a^b J_0(x)\,dx + \delta\big(J_0(a)+J_0(b)\big)$$
$$+\frac{p}{a\pi}\left(\frac{\delta}{a-\delta}+1\right)\sin a\pi+\frac{p}{b\pi}\left(\frac{\delta}{b+\delta}-1\right)\sin b\pi-\frac{p\delta}{a-\delta}\cos a\pi-\frac{p\delta}{b+\delta}\cos b\pi$$

Thus

$$\int_{a-\delta}^{b+\delta} J_p(x)dx \simeq \int_a^b J_0(x)dx + \delta\big(J_0(a)+J_0(b)\big)$$
$$+\frac{p\sin a\pi}{\pi(a-\delta)}-\frac{p\sin b\pi}{\pi(b+\delta)}-\frac{p\delta\cos a\pi}{a-\delta}-\frac{p\delta\cos b\pi}{b+\delta}$$

Theorem 2.12: If a , b are unlimited such that $a \simeq b+\delta$, for $\delta \in \text{mon}(0)$, then

$$\int_a^b J_p(x)dx = \int_\omega^{\omega+\delta} J_p(x)dx \simeq \delta J_0\left(\omega+\tfrac{\delta}{2}\right)+\frac{p\delta\pi}{(\omega+\delta)}\sqrt{\frac{\omega}{2}}J_{\frac{3}{2}}(\omega\pi)$$

For $a=\omega$ is unlimited and $\delta \in \text{mon}(0)$.

Proof:

Since a , b are unlimited such that $a \in \text{mon}(b)$ where $a=\omega$ and $b=\omega+\delta$ for ω is unlimited and $\delta \in \text{mon}(0)$ then $b-a=\delta \simeq 0$, and by using Lemma (2.1) and Theorem (2.5), we get

$$\int_\omega^{\omega+\delta} J_p(x)dx \simeq \frac{4}{\pi}\int_0^1 \frac{\sin\left(\frac{\omega+\delta-\omega}{2}t\right)\cos\left(\frac{\omega+\delta+\omega}{2}t\right)}{t\sqrt{1-t^2}}dt+\frac{p\sin\pi\omega}{\pi\omega}-\frac{p\sin(\omega+\delta)\pi}{\pi(\omega+\delta)}$$

Since $\delta \in \text{mon}(0)$, then $\sin\frac{\delta}{2}t \simeq \frac{\delta}{2}t$, $\cos\delta\pi \simeq 1$ and $\sin\delta\pi \simeq \delta\pi$, so

$$\sin(\omega+\delta)\pi = \sin\pi\omega + \delta\pi\cos\omega\pi$$

Therefore

$$\int_\omega^{\omega+\delta} J_p(x)dx \simeq \frac{2\delta}{\pi}\int_0^1 \frac{\cos\left(\left(\omega+\frac{\delta}{2}\right)t\right)}{\sqrt{1-t^2}}dt+\frac{p\sin\omega\pi}{\omega\pi}-\frac{p(\sin\pi\omega+\delta\pi\cos\omega\pi)}{\pi(\omega+\delta)}$$
$$\simeq \delta J_0\left(\omega+\frac{\delta}{2}\right)+\frac{p}{\pi}\left(\frac{1}{\omega}-\frac{1}{\omega+\delta}\right)\sin\omega\pi-\frac{p\delta}{\omega+\delta}\cos\omega\pi$$
$$\simeq \delta J_0\left(\omega+\frac{\delta}{2}\right)+\frac{p\delta}{\omega+\delta}\left(\frac{\sin\omega\pi}{\omega\pi}-\cos\omega\pi\right)$$

Using Proposition (2.8) we get

$$\int_\omega^{\omega+\delta} J_p(x)dx \simeq \delta J_0\left(\omega+\tfrac{\delta}{2}\right)+\frac{p\delta\pi}{(\omega+\delta)}\sqrt{\frac{\omega}{2}}J_{\frac{3}{2}}(\omega\pi)$$

Theorem 2.13: If a , b are unlimited such that $a \in \text{gal}(b)$, then

$$\left|\int_a^{a+r} J_p(x)dx\right| \le r+\frac{p}{\pi}(\delta+\varepsilon), \quad \text{for } \varepsilon, \delta \in \text{mon}(0) \text{ and } r, p \text{ are limited.}$$

Proof:

Since a , b are unlimited such that $a \in gal(b)$, then by using Theorem (2.5) we get

$$\int_a^{a+r} J_p(x)\,dx \simeq \int_a^{a+r} J_0(x)\,dx + p\left(\frac{\sin a\pi}{a\pi} - \frac{\sin(a+r)\pi}{(a+r)\pi}\right)$$

Thus

$$\left|\int_a^{a+r} J_p(x)\,dx\right| \le \left|\int_a^{a+r} J_0(x)\,dx\right| + \frac{p}{a\pi}|\sin a\pi| + \frac{p}{\pi(a+r)}|\sin(a+r)\pi|$$

Since $|\sin x| \le 1$, then

$$\left|\int_a^{a+r} J_p(x)\,dx\right| \le \int_a^{a+r} |J_0(x)|\,dx + \frac{p}{a\pi} + \frac{p}{\pi(a+r)}$$

Since $|J_0(x)| \le 1$, then

$$\int_a^{a+r} |J_0(x)|\,dx \le \int_a^{a+r} dx = r$$

Now for unlimited a, we have so $\frac{1}{a} \simeq \delta$ and $\frac{1}{a+r} \simeq \varepsilon$ where $\varepsilon\,, \delta \in mon(0)$ then

$$\left|\int_a^{a+r} J_p\,(x)dx\right| \le r + \frac{p}{\pi}(\delta + \varepsilon)$$

Theorem 2.14: If a , b are unlimited such that $a = \omega\,, b = 2\omega$ where ω is unlimited then

$$\int_\omega^{2\omega} J_p\,(x)dx \le \int_\omega^{2\omega} J_0\,(x)dx + 2p\delta$$

For $\delta \in mon(0)$ and p is any real value.

Proof:

Since by applying Theorem (2.5) for the assumed values a and b, we get

$$\int_\omega^{2\omega} J_p(x)\,dx \simeq \int_\omega^{2\omega} J_0(x)\,dx + p\left(\frac{\sin\omega\pi}{\omega\pi} - \frac{\sin 2\omega\pi}{2\omega\pi}\right) \simeq \int_\omega^{2\omega} J_0(x)\,dx + \frac{p}{\omega\pi}\sin\omega\pi - \frac{p}{\omega\pi}\sin\omega\pi\cos\omega\pi$$

$$\simeq \int_\omega^{2\omega} J_0(x)\,dx + 2p\sin^2\tfrac{\omega\pi}{2}\cdot\frac{\sin\omega\pi}{\omega\pi}$$

Since ω is unlimited then $\frac{\sin\omega\pi}{\omega\pi} = \delta \simeq 0$ and $\sin^2\frac{\omega\pi}{2} \le 1$ then

$$\int_\omega^{2\omega} J_p\,(x)dx \le \int_\omega^{2\omega} J_0\,(x)dx + 2p\delta$$

Theorem 2.15: Let I=[a,b]. Then the integral of Bessel function for p=0, on I is given by the series expansion

$$\int_a^b J_0\,(x)dx = \sum_{k=0}^{\omega} \frac{(-1)^k (b^{2k+1} - a^{2k+1})}{(2k+1)2^{2k}(k!)^2},$$

where ω is unlimited.

Proof:

By expanding $\sin x$ in Equation (2.1.2) using Taylor series up to unlimited $n = \omega$, we get

$$\int_a^b J_0(x)\,dx = \frac{2}{\pi}\int_0^1 \frac{\sin(bt) - \sin(at)}{t\sqrt{1-t^2}}dt = \frac{2}{\pi}\sum_{k=0}^{\omega} \frac{(-1)^k (b^{2k+1} - a^{2k+1})}{(2k+1)!}\int_0^1 \frac{t^{2k}dt}{\sqrt{1-t^2}}$$

$$= \frac{2}{\pi}\sum_{k=0}^{\omega} \frac{(-1)^k (b^{2k+1} - a^{2k+1})}{(2k+1)!}\int_0^1 (t^2)^k\,(1-t^2)^{\frac{-1}{2}}dt$$

Let $u = t^2$ then $dt = \frac{du}{2\sqrt{u}}$. Therefore

$$\int_a^b J_0(x)\,dx = \frac{1}{\pi}\sum_{k=0}^{\omega}\frac{(-1)^k(b^{2k+1}-a^{2k+1})}{(2k+1)!}\int_0^1 (u)^{k+\frac{1}{2}-1}(1-u)^{\frac{1}{2}-1}dt$$

and

$$\int_a^b J_0(x)\,dx = \frac{1}{\pi}\sum_{k=0}^{\omega}\frac{(-1)^k(b^{2k+1}-a^{2k+1})}{(2k+1)!}\cdot\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(k+1)}$$

Since

$$\Gamma\left(k+\frac{1}{2}\right) = \frac{(2k)!\sqrt{\pi}}{2^{2k}k!},\quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\quad,\ \Gamma(k+1) = k!$$

Then

$$\int_a^b J_0(x)\,dx = \frac{1}{\pi}\sum_{k=0}^{\omega}\frac{(-1)^k(b^{2k+1}-a^{2k+1})}{(2k+1)(2k)!}\cdot\frac{\pi(2k)!}{(k!)^2 2^{2k}} = \sum_{k=0}^{\omega}\frac{(-1)^k(b^{2k+1}-a^{2k+1})}{(2k+1)2^{2k}(k!)^2}$$

where ω is unlimited.

Theorem 2.16: Let I=[a,b]. If $a \simeq b$, then

1) $\int_a^b J_0(x)dx \simeq \delta J_0(a)$ 2) $\int_a^b J_0(x)dx \simeq \frac{2}{\pi}\delta \quad , \quad \delta \simeq 0$

Proof:

1) Since $a \simeq b$ then $2a \simeq b + a$, and $\sin\frac{\delta t}{2} \simeq \frac{\delta t}{2}$ then by using Lemma (2.1) we get

$$\int_a^b J_0(x)\,dx = \frac{4}{\pi}\int_0^1\frac{\sin\left(\frac{b-a}{2}t\right)\cos\left(\frac{b+a}{2}t\right)}{t\sqrt{1-t^2}}dt \simeq \frac{2\delta}{\pi}\int_0^1\frac{\cos at}{\sqrt{1-t^2}}dt = \delta J_0(a)$$

2) since $a \simeq b$ then $at \simeq bt$ where $0 < t < 1$ and $\sin at \simeq \sin bt$ then

$$\frac{\sin at}{t\sqrt{1-t^2}} \simeq \frac{\sin bt}{t\sqrt{1-t^2}} \text{ and } \frac{\sin at}{t\sqrt{1-t^2}} - \frac{\sin bt}{t\sqrt{1-t^2}} \simeq \delta \simeq 0$$

Thus

$$\frac{2}{\pi}\int_0^1\frac{\sin at - \sin bt}{t\sqrt{1-t^2}}dt \simeq \frac{2}{\pi}\delta$$

So by using Equation (2.1.2) we get

$$\int_a^b J_0(x)dx \simeq \frac{2}{\pi}\delta.$$

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