International Journal of Enhanced Research in Science, Technology & Engineering
ISSN: 2319-7463, Vol. 5 Issue 4, April-2016

Flow through Layered Media with Embedded Transition Porous Layer

M. S. Abu Zaytoon[1], T. L. Alderson[2], M. H. Hamdan[3]

[1,2,3] Dept. of Mathematical Sciences, University of New Brunswick, P.O. Box 5050, Saint John, New Brunswick, CANADA E2L 4L5

ABSTRACT

Flow through a composite of three porous layers is considered. The middle layer is assumed to be of variable permeability and the bounding layers are of constant permeability. Flow through the layers is governed by Brinkman's equation which, for the chosen configuration, is reduced to an Airy's differential equation valid in the variable permeability layer. Exact solutions are obtained and extensive computations are provided to evaluate Airy's functions and the Nield-Kuznetsov function.

Keywords: Airy's Functions, Layered Media, Transition Layer.

1. INTRODUCTION

Vafai and Thiyagaraja, [1], identified the flow through one porous layer over another porous layer as the second problem in the flow over porous layers. This problem has been less studied than flow through a Navier-Stokes channel over a porous layer, which has received extensive analysis over the last five decades due to the need to derive appropriate interfacial conditions and to determine the most appropriate flow model to use in describing the flow through the porous layer, [2], [3].

Flow through composite porous layers (defined here as porous layers of differing porosities and permeabilities), is of interest due to its many applications that arise in both natural settings and in biological and industrial applications, [4], [5], [6]. Of particular interest in these applications is the flow through variable permeability porous layers, due to the natural occurrence of non-constant permeability porous layers such as earth layers where oil and water recovery are of great importance, [6].

Some studies have already been devoted to flow through variable permeability media, [6], [7], [8]. In fact, in a recent article, Nield and Kuznetzov [9] presented elegant analysis of flow over a porous layer based on a formulation in which they considered the flow through a transition Brinkman layer, [10], [11], of variable thickness and variable permeability bounded by a constant permeability Brinkman layer, on one side, and a free-space channel on the other. They expressed their solution in terms of Airy's functions and introduced an important new function, $N_i(x)$, that proved to be of great utility in the solution of inhomogeneous Airy's ODE with a variable forcing function. Hamdan and Kamel [12] studied extensively the $N_i(x)$, and introduced a new function, $K_i(x)$, that complements the $N_i(x)$ function in obtaining the solution to inhomogeneous Airy's ODE. The pair of functions $N_i(x)$ and $K_i(x)$ are known as the Nield-Kuznetsov functions.

The approach used by Nield and Kuznetsov [9] in formulating and solving the transition layer problem proved to be of great utility in solving similar practical problems, [11]. Their approach is also well-suited in the general analysis of flow through composite porous layers of variable thicknesses and permeabilities, which is the subject matter of this current work. We consider the flow through a Brinkman porous layer that is bounded by two constant permeability Brinkman layers. The bounding lower and upper layers are themselves bounded by impermeable, macroscopic boundaries on which the no-slip condition is imposed. The flow through each bounding layer is assumed to be governed by Brinkman's equation with a different, yet constant permeability for each layer. The middle layer is governed by Brinkman's equation with variable permeability. At the interfaces between layers, it is assumed that permeability is continuous. Other conditions at the interface between layers are velocity continuity and shear stress continuity. The objectives here are to derive expressions for velocity and shear stress at the interface, and to determine the velocity profiles in the layers. Velocity in the variable permeability layer is expressed and evaluated in terms of the Nield-Kuznetsov function. The derived velocity expressions are compatible with the equations governing flow through two-dimensional channels, hence should serve as velocity entry conditions to flow through two-dimensional configurations.

2. PROBLEM FORMULATION

Consider the flow through three porous layers of different thicknesses, and of different permeability, shown in **Fig.1**. The middle layer is of variable permeability, while the bounding lower and upper layers are of constant permeability. Each of the bounding layers is terminated by a solid wall on their outer sides and join the middle variable-permeability layer along an assumed sharp interface. The flow is assumed to be driven by the same pressure gradient and governed by Brinkman's equation in each layer.

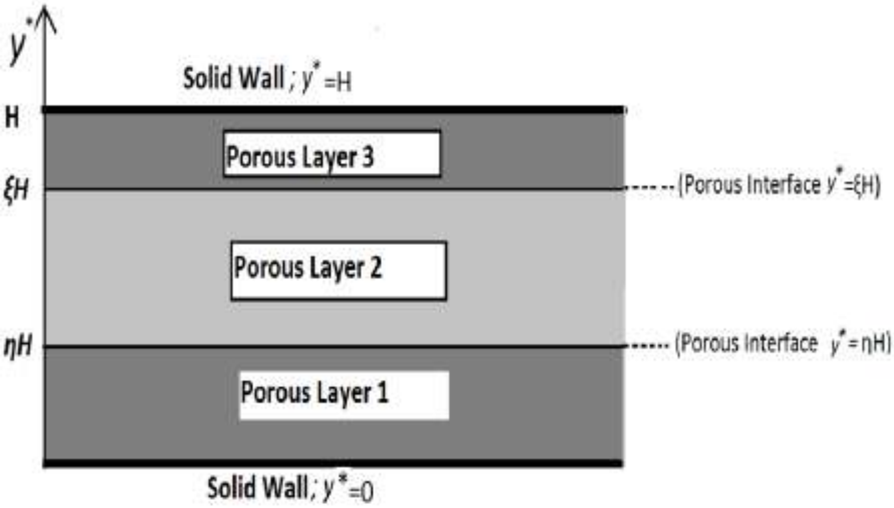

Figure 1. Representative Sketch

In layer 1: the permeability, K_1, in layer 1 is assumed constant and given by:

$$K_1 = aK_0; 0 < y^* < \eta H. \qquad \ldots(1)$$

In layer 2: the permeability, K_2, in layer 2 is assumed to be a variable function of y^*, and given by:

$$K_2(y^*) = \frac{abK_0(\eta - \xi)H}{(b-a)y^* + (a\eta - b\xi)H}; \eta H < y^* < \xi H. \qquad \ldots(2)$$

In layer 3: the permeability, K_3, in layer 3 is assumed constant and given by:

$$K_3 = bK_0; \xi H < y^* < H. \qquad \ldots(3)$$

In (1), (2) and (3), K_0 is a reference constant permeability, a and b are constants to be selected, ξ and η are parameters that determine the thickness of each layer. We note that at each interface between layers the permeability is continuous with $K_1 = K_2(\eta H) = aK_0$ and $K_3 = K_2(\xi H) = bK_0$.

The governing equations and boundary conditions associated with the configuration in **Figure 1** are stated as follows.

In layer 1, the flow is governed by:

$$\mu_{1eff} \frac{d^2 u^*_1}{dy^{*2}} - \frac{\mu_1}{K_1} u^*_1 + G = 0 ; \quad 0 < y^* < \eta H. \qquad \ldots(4)$$

In layer 2, the flow is governed by:

$$\mu_{2eff}\frac{d^2u^*_2}{dy^{*2}}-\frac{\mu_2}{K_2(y^*)}u^*_2+G=0\ ;\ \eta H<y^*<\xi H. \qquad \text{…(5)}$$

In layer 3, the flow is governed by:

$$\mu_{3eff}\frac{d^2u^*_3}{dy^{*2}}-\frac{\mu_3}{K_3}u^*_3+G=0\ ;\ \xi H<y^*<H. \qquad \text{…(6)}$$

In (4), (5) and (6), $G=-\frac{dp}{dx}$ is the constant pressure gradient, $u^*_i=u^*_i(y^*)$ is the velocity in the ith layer, for i=1,2,3, K_i is the permeability in the ith layer, μ_i is the viscosity of the base-fluid saturating the ith layer, and μ_{ieff} is the effective viscosity of the fluid in the ith layer. We note that the viscosities of the base fluids in the three layers should be equal, $\mu_1=\mu_2=\mu_3$, if the base fluid is the same. The effective viscosities, μ_{1eff}, μ_{2eff}, and μ_{3eff}, however, are not necessarily equal.

Now, introducing the dimensionless variables:

$$y=\frac{y^*}{H};u_i=\frac{\mu_i}{GH^2}u^*_i\ ;\ M_i=\frac{\mu_{ieff}}{\mu_i} \qquad \text{…(7)}$$

and defining $Da=\frac{K_0}{H^2}$ as the Darcy number, and (4), (5) and (6) take the following dimensionless forms, respectively:

$$M_1\frac{d^2u_1}{dy^2}-\frac{H^2}{K_1}u_1+1=0\ ;\quad 0<y<\eta. \qquad \text{…(8)}$$

$$M_2\frac{d^2u_2}{dy^2}-\frac{H^2}{K_2(y)}u_2+1=0\ ;\quad \eta<y<\xi. \qquad \text{…(9)}$$

$$M_3\frac{d^2u_3}{dy^2}-\frac{H^2}{K_3}u_3+1=0\ ;\quad \xi<y<1. \qquad \text{…(10)}$$

Upon using the permeability distributions (1), (2), and (3) in (8), (9), and (10), respectively, we obtain:

$$\frac{d^2u_1}{dy^2}-\frac{1}{aDaM_1}u_1+\frac{1}{M_1}=0\ ;\quad 0<y<\eta. \qquad \text{…(11)}$$

$$\frac{d^2u_2}{dy^2}-\frac{(b-a)y+a\eta-b\xi}{abDaM_2(\eta-\xi)}u_2+\frac{1}{M_2}=0\ ;\quad \eta<y<\xi. \qquad \text{…(12)}$$

$$\frac{d^2u_3}{dy^2}-\frac{1}{bDaM_3}u_3+\frac{1}{M_3}=0\ ;\quad \xi<y<1. \qquad \text{…(13)}$$

Equations (11), (12) and (13) are to be solved subject to the conditions of no-slip at the solid walls (y = 0 and y = 1), velocity and shear-stress continuity at the interfaces between layers ($y=\eta$ and $y=\xi$), where the shear stress in each layer is defined based on the effective viscosity, namely $\mu_{ieff}\frac{du_i}{dy}$ for i=1,2,3. These conditions are stated as follows:

$$u_1(0) = u_3(1) = 0\,;\; u_1(\eta) = u_2(\eta)\; u_2(\xi) = u_3(\xi) \qquad \ldots(14)$$

$$\frac{du_1}{dy}(\eta) = \vartheta_1 \frac{du_2}{dy}(\eta)\,;\; \vartheta_2 \frac{du_2}{dy}(\xi) = \frac{du_3}{dy}(\xi)\,;\; \vartheta_1 = \frac{\mu_{2eff}}{\mu_{1eff}}\,;\; \vartheta_2 = \frac{\mu_{2eff}}{\mu_{3eff}}\,. \qquad \ldots(15)$$

3. METHOD OF SOLUTION

3.1. Solving the Governing Equations

Defining

$$\lambda_1 = \frac{1}{\sqrt{aDaM_1}},\; \lambda_2 = \frac{1}{\sqrt[3]{ab(b-a)^2 DaM_2(\eta-\xi)}},\; \lambda_3 = \frac{1}{\sqrt{bDaM_3}} \qquad \ldots(16)$$

then (11), (12) and (13) can be written, respectively, as:

$$\frac{d^2u_1}{dy^2} - \lambda^2{}_1 u_1 + \frac{1}{M_1} = 0 \;;\quad 0 < y < \eta. \qquad \ldots(17)$$

$$\frac{d^2u_2}{dy^2} - \lambda^3{}_2 (b-a)^2[(b-a)y + a\eta - b\xi]u_2 + \frac{1}{M_2} = 0 \;;\quad \eta < y < \xi. \qquad \ldots(18)$$

$$\frac{d^2u_3}{dy^2} - \lambda^2{}_3 u_3 + \frac{1}{M_3} = 0 \;;\quad \xi < y < 1. \qquad \ldots(19)$$

Now, (17) and (19) are linear, inhomogeneous, second-order ODE whose general solutions are given, respectively, by:

$$u_1(y) = a_1 \exp(\lambda_1 y) + a_2 \exp(-\lambda_1 y) + \frac{1}{M_1 \lambda^2{}_1} \qquad \ldots(20)$$

$$u_3(y) = c_1 \exp(\lambda_3 y) + c_2 \exp(-\lambda_3 y) + \frac{1}{M_2 \lambda^2{}_3} \qquad \ldots(21)$$

where a_1, a_2, c_1 and c_2 are arbitrary constants to be determined using boundary and interface conditions (14) and (15).

Shear stress across the first and third layers is given by the first derivative of each of (20) and (21), namely

$$\frac{du_1}{dy} = a_1\lambda_1 \exp(\lambda_1 y) - a_2\lambda_1 \exp(-\lambda_1 y) \qquad \ldots(22)$$

$$\frac{du_3}{dy} = c_1\lambda_3 \exp(\lambda_3 y) - c_2\lambda_3 \exp(-\lambda_3 y). \qquad \ldots(23)$$

In order to solve (18), we first introduce the transformation:

$$Y = \lambda_2[(b-a)y + a\eta - b\xi]\,;\;\; u_2(y) \equiv U_2(Y)\,. \qquad \ldots(24)$$

Equation (18) then takes the form:

$$\frac{d^2U_2}{dY^2} - YU_2 + \frac{1}{\lambda^2{}_2 (b-a)^2 M_2} = 0 \;;\quad b\lambda_2(\eta-\xi) < Y < a\lambda_2(\eta-\xi) \qquad \ldots(25)$$

which is Airy's inhomogeneous differential equation. Solution to the homogeneous part of (25) is given in terms of Airy's functions, $A_i(Y)$ and $B_i(Y)$, [13], and the complementary function is given by the linear combination:

$$U_{2c}(Y) = b_1 A_i(Y) + b_2 B_i(Y) \qquad \ldots(26)$$

where b_1 and b_2 are arbitrary constants. The Wronskian of $A_i(Y)$ and $B_i(Y)$ is given by, [13]:

$$W(A_i(Y), B_i(Y)) = A_i(Y)B_i'(Y) - A_i'(Y)B_i(Y) = \frac{1}{\pi} \qquad \ldots(27)$$

where prime notation denotes ordinary differentiation with respect to the argument. The particular integral of (32) can thus be obtained by the method of variation of parameters, and takes the form:

$$U_{2p} = \frac{\pi}{\lambda^2{}_2(b-a)^2 M_2}\left[A_i(Y)\int_0^Y B_i(t)dt - B_i(Y)\int_0^Y A_i(t)dt\right]. \qquad \ldots(28)$$

The expression in brackets on the RHS of (28), and its derivative, namely

$$N_i(Y) = A_i(Y)\int_0^Y B_i(t)dt - B_i(Y)\int_0^Y A_i(t)dt \qquad \ldots(29)$$

$$N_i'(Y) = A_i'(Y)\int_0^Y B_i(t)dt - B_i'(Y)\int_0^Y A_i(t)dt \qquad \ldots(30)$$

are recognized as the Nield-Kuznetsov function and its derivative, and has been studied extensively by Hamdan and Kamel [12]. Upon using (26), (28) and (29), we express the general solution to (25) as:

$$U_2(Y) = b_1 A_i(Y) + b_2 B_i(Y) + \frac{\pi}{\lambda^2{}_2(b-a)^2 M_2} N_i(Y); \qquad b\lambda_2(\eta-\xi) < Y < a\lambda_2(\eta-\xi). \qquad \ldots(31)$$

Using (24) in (31), we obtain:

$$u_2(y) = b_1 A_i\big(\lambda_2[(b-a)y + a\eta - b\xi]\big) + b_2 B_i\big(\lambda_2[(b-a)y + a\eta - b\xi]\big)$$
$$+ \frac{\pi}{\lambda^2{}_2(b-a)^2 M_2} N_i\big(\lambda_2[(b-a)y + a\eta - b\xi]\big) \qquad ; \qquad \eta < y < \xi. \qquad \ldots(32)$$

Derivative of (32) with respect to y is the shear stress term across the second layer, and is given by:

$$\frac{du_2}{dy} = b_1\lambda_2(b-a)A_i'\big(\lambda_2[(b-a)y + a\eta - b\xi]\big) + b_2\lambda_2(b-a)B_i'\big(\lambda_2[(b-a)y + a\eta - b\xi]\big)$$
$$+ \frac{\pi}{\lambda_2(b-a)M_2} N_i'\big(\lambda_2[(b-a)y + a\eta - b\xi]\big); \eta < y < \xi. \qquad \ldots(33)$$

3.2. Determining the Arbitrary Constants

Equations (20), (21) and (32) represent the general solutions to the governing equations (11), (12) and (13). Using conditions (14) and (15) in (20)-(23) and (32)-(33), we obtain the following system of linear equations for the arbitrary constants appearing in the velocity equations, written in the matrix-vector form

$$MX = C$$
…(34)

where

$$X=\begin{bmatrix} a_1\\ a_2\\ b_1\\ b_2\\ c_1\\ c_2 \end{bmatrix};\qquad C=\begin{bmatrix} -\dfrac{1}{M_1\lambda^2{}_1}\\ \dfrac{\pi N_i(\lambda_2 b[\eta-\xi])}{\lambda^2{}_2(b-a)^2M_2}-\dfrac{1}{M_1\lambda^2{}_1}\\ \dfrac{\pi N_i(\lambda_2 a[\eta-\xi])}{\lambda^2{}_2(b-a)^2M_2}-\dfrac{1}{M_2\lambda^2{}_3}\\ \dfrac{\vartheta_1\pi N_i'(\lambda_2 b[\eta-\xi])}{\lambda_2(b-a)M_2}\\ \dfrac{\vartheta_2\pi N_i'(\lambda_2 a[\eta-\xi])}{\lambda_2(b-a)M_2}\\ -\dfrac{1}{M_2\lambda^2{}_3} \end{bmatrix};$$

$$M=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0\\ \exp(\lambda_1\eta) & \exp(-\lambda_1\eta) & -A_i(\lambda_2 b[\eta-\xi]) & -B_i(\lambda_2 b[\eta-\xi]) & 0 & 0\\ 0 & 0 & -A_i(\lambda_2 a[\eta-\xi]) & -B_i(\lambda_2 a[\eta-\xi]) & \exp(\lambda_3\xi) & \exp(-\lambda_3\xi)\\ \lambda_1\exp(\lambda_1\eta) & -\lambda_1\exp(-\lambda_1\eta) & -\vartheta_1\lambda_2(b-a)A_i'(\lambda_2 b[\eta-\xi]) & -\vartheta_1\lambda_2(b-a)B_i'(\lambda_2 b[\eta-\xi]) & 0 & 0\\ 0 & 0 & -\vartheta_2\lambda_2(b-a)A_i'(\lambda_2 a[\eta-\xi]) & -\vartheta_2\lambda_2(b-a)B_i'(\lambda_2 a[\eta-\xi]) & \lambda_3\exp(\lambda_3\xi) & -\lambda_3\exp(-\lambda_3\xi)\\ 0 & 0 & 0 & 0 & \exp(\lambda_3) & \exp(-\lambda_3) \end{bmatrix}$$

Solution to (34) requires evaluations functions of Airy's functions, their derivatives and integrals. This is discussed in what follows.

3.3. Asymptotic Approximations of the Functions A_i, B_i and N_i

The velocity profile and shear stress term in the variable permeability Brinkman layer involve Airy's functions and their first derivatives, and the Nield-Kuznetsov function and its first derivative. These functions will be evaluated at their respective arguments using asymptotic approximations. We find it convenient to first introduce the following acronyms:

$$\gamma_1=\lambda_2(b-a),\ \gamma_2=\lambda_2(a\eta-b\xi),\ \gamma_3=\frac{\pi}{\lambda^2{}_2(b-a)^2M_2},\ \gamma_4=\frac{\pi}{\lambda_2(b-a)M_2}. \qquad \text{…(35)}$$

Using (35), equations (32) and (33) can be written, respectively, as

$$u_2(y)=b_1A_i(\gamma_1y+\gamma_2)+(b_2-\frac{\gamma_3}{3})B_i(\gamma_1y+\gamma_2);\qquad \eta<y<\xi. \qquad \text{…(36)}$$

$$\frac{du_2}{dy}=b_1\lambda_2(b-a)A_i'(\gamma_1y+\gamma_2)+[b_2\lambda_2(b-a)-\frac{\gamma_4}{3}]B_i'(\gamma_1y+\gamma_2);\quad \eta<y<\xi. \qquad \text{…(37)}$$

The values of γ_1 and γ_2 are dependent on λ_2, which is a function of Da,ξ,η,a,b and M_2, as shown in (16) and (35). For a given variable porous layer thickness, $\xi-\eta$, and permeability parameters, a and b, the values of λ_2 are sensitive to variations in Darcy number, Da. For small values of Da, the value of λ_2 is large, and for large arguments of Airy's and Nield-Kuznetsov function, we can use the asymptotic approximations in **Table 1**, [12]:

Table 1. Asymptotic Approximations

$A_i(Y) \approx \frac{\exp(-2Y^{3/2}/3)}{2\sqrt{\pi}Y^{1/4}}$	$A_i'(Y) \approx -\frac{Y^{1/4}\exp(-2Y^{3/2}/3)}{2\sqrt{\pi}}$	$\int_0^Y A_i(t)dt \approx \frac{1}{3}$
$B_i(Y) \approx \frac{\exp(2Y^{3/2}/3)}{\sqrt{\pi}Y^{1/4}}$	$B_i'(Y) \approx -\frac{Y^{1/4}\exp(2Y^{3/2}/3)}{\sqrt{\pi}}$	$\int_0^Y B_i(t)dt \approx \frac{\exp(2Y^{3/2}/3)}{\sqrt{\pi}Y^{3/4}}$
$N_i(Y) \approx -\frac{B_i(Y)}{3}$	$N_i'(Y) \approx -\frac{B_i'(Y)}{3}$	$\int_0^Y N_i(t)dt \approx -\frac{1}{3}\int_0^Y B_i(t)dt$

3.4. Calculating the Mean Velocity through the Three Layers

Mean velocity across the channel layers is defined as

$$\bar{u} = \bar{u}_1 + \bar{u}_2 + \bar{u}_3 = \int_0^\eta u_1 dy + \int_\eta^\xi u_2 dy + \int_\xi^1 u_3 dy \qquad \ldots(38)$$

where $\bar{u}_1, \bar{u}_2, \bar{u}_3$ are the mean velocities in layers 1, 2, and 3 respectively.

In order to utilize the asymptotic approximations of **Table 1** in (38), the middle integral in (38) must be converted to one with a lower bound of zero. We first write the integral as:

$$\int_\eta^\xi u_2 dy = \int_0^\xi u_2 dy - \int_0^\eta u_2 dy \,. \qquad \ldots(39)$$

Using **Table 1** in (39), we obtain

$$\begin{aligned}\int_\eta^\xi u_2 dy = &\int_0^\xi [b_1 A_i(\gamma_1 y + \gamma_2) + b_2 B_i(\gamma_1 y + \gamma_2) + \gamma_3 N_i(\gamma_1 y + \gamma_2)]dy - \\ &\int_0^\eta [b_1 A_i(\gamma_1 y + \gamma_2) + b_2 B_i(\gamma_1 y + \gamma_2) + \gamma_3 N_i(\gamma_1 y + \gamma_2)]dy\end{aligned} \qquad \ldots(40)$$

Letting $t = (\gamma_1 y + \gamma_2)$ we obtain $dt = \gamma_1 dy$, and $(\gamma_1 \eta + \gamma_2) \le t \le (\gamma_1 \xi + \gamma_2)$. We can then write (40) as

$$\int_\eta^\xi u_2 dy = \frac{b_1}{\gamma_1}\int_{(\gamma_1\eta+\gamma_2)}^{(\gamma_1\xi+\gamma_2)} A_i(t)dt + \frac{b_2}{\gamma_1}\int_{(\gamma_1\eta+\gamma_2)}^{(\gamma_1\xi+\gamma_2)} B_i(t)dt + \frac{\gamma_3}{\gamma_1}\int_{(\gamma_1\eta+\gamma_2)}^{(\gamma_1\xi+\gamma_2)} N_i(t)dt \qquad \ldots(41)$$

where

$$\int_{(\gamma_1\eta+\gamma_2)}^{(\gamma_1\xi+\gamma_2)} A_i(t)dt = \int_{(\gamma_1\eta+\gamma_2)}^{0} A_i(t)dt + \int_0^{(\gamma_1\xi+\gamma_2)} A_i(t)dt = \int_0^{(\gamma_1\xi+\gamma_2)} A_i(t)dt - \int_0^{(\gamma_1\eta+\gamma_2)} A_i(t)dt = 0 \qquad \ldots(42)$$

$$\begin{aligned}\int_{(\gamma_1\eta+\gamma_2)}^{(\gamma_1\xi+\gamma_2)} B_i(t)dt &= \int_{(\gamma_1\eta+\gamma_2)}^{0} B_i(t)dt + \int_0^{(\gamma_1\xi+\gamma_2)} B_i(t)dt = \int_0^{(\gamma_1\xi+\gamma_2)} B_i(t)dt - \int_0^{(\gamma_1\eta+\gamma_2)} B_i(t)dt \\ &= \frac{\exp(2(\gamma_1\xi+\gamma_2)^{3/2}/3)}{\sqrt{\pi}(\gamma_1\xi+\gamma_2)^{7/4}} - \frac{\exp(2(\gamma_1\eta+\gamma_2)^{3/2}/3)}{\sqrt{\pi}(\gamma_1\eta+\gamma_2)^{7/4}}\end{aligned} \qquad \ldots(43)$$

$$\int_{(\gamma_1\eta+\gamma_2)}^{(\gamma_1\xi+\gamma_2)} N_i(t)dt = \int_0^{(\gamma_1\xi+\gamma_2)} N_i(t)dt - \int_0^{(\gamma_1\eta+\gamma_2)} N_i(t)dt = -\frac{1}{3}\int_0^{(\gamma_1\xi+\gamma_2)} B_i(t)dt + \frac{1}{3}\int_0^{(\gamma_1\eta+\gamma_2)} B_i(t)dt$$
$$= -\frac{1}{3}\frac{\exp(2(\gamma_1\xi+\gamma_2)^{3/2}/3)}{\sqrt{\pi}(\gamma_1\xi+\gamma_2)^{7/4}} + \frac{1}{3}\frac{\exp(2(\gamma_1\eta+\gamma_2)^{3/2}/3)}{\sqrt{\pi}(\gamma_1\eta+\gamma_2)^{7/4}} \quad \ldots(44)$$

We thus have

$$\bar{u}_2 = \int_\eta^\xi u_2 dy = [\frac{b_2}{\gamma_1} - \frac{\gamma_3}{3\gamma_1}]\left[\frac{\exp(2(\gamma_1\xi+\gamma_2)^{3/2}/3)}{\sqrt{\pi}(\gamma_1\xi+\gamma_2)^{7/4}} - \frac{\exp(2(\gamma_1\eta+\gamma_2)^{3/2}/3)}{\sqrt{\pi}(\gamma_1\eta+\gamma_2)^{7/4}}\right]. \quad \ldots(45)$$

Other integrals appearing in the mean velocity, (38), are evaluated using (22) and (23), as follows:

$$\bar{u}_1 = \int_0^\eta u_1 dy = \int_0^\eta [a_1\exp(\lambda_1 y) + a_2\exp(-\lambda_1 y) + \frac{1}{M_1\lambda^2{}_1}]dy = \frac{1}{\lambda_1}[a_1\{\exp(\lambda_1\eta)-1\} - a_2\{\exp(-\lambda_1\eta)-1\} + \frac{\eta}{M_1\lambda_1}] \quad \ldots(46)$$

$$\bar{u}_3 = \int_\xi^1 u_3 dy = \int_\xi^1 [c_1\exp(\lambda_3 y) + c_2\exp(-\lambda_3 y) + \frac{1}{M_2\lambda^2{}_3}]dy$$
$$= \frac{1}{\lambda_3}\left[c_1\{\exp(\lambda_3) - \exp(\lambda_3\xi)\} - c_2\{\exp(-\lambda_3) - \exp(-\lambda_3\xi)\} + \frac{1-\xi}{M_2\lambda_3}\right]. \quad \ldots(47)$$

Upon using (45), (48) and (49) in (38) we obtain the following expression for mean velocity across the flow domain:

$$\bar{u} = \frac{1}{\lambda_1}[a_1\{\exp(\lambda_1\eta)-1\} - a_2\{\exp(-\lambda_1\eta)-1\} + \frac{\eta}{M_1\lambda_1}]$$
$$+\frac{1}{\lambda_3}\left[c_1\{\exp(\lambda_3) - \exp(\lambda_3\xi)\} - c_2\{\exp(-\lambda_3) - \exp(-\lambda_3\xi)\} + \frac{1-\xi}{M_2\lambda_3}\right]. \quad \ldots(48)$$
$$+[\frac{b_2}{\gamma_1} - \frac{\gamma_3}{3\gamma_1}]\left[\frac{\exp(2(\gamma_1\xi+\gamma_2)^{3/2}/3)}{\sqrt{\pi}(\gamma_1\xi+\gamma_2)^{7/4}} - \frac{\exp(2(\gamma_1\eta+\gamma_2)^{3/2}/3)}{\sqrt{\pi}(\gamma_1\eta+\gamma_2)^{7/4}}\right]$$

4. RESULTS AND DISCUSSION

4.1. Choice of Permeability Parameters, Layer Thickness and Effective Viscosities

The constant permeability control parameters a and b influence the choice of the variable permeability in the middle layer, as can be seen from equation (2). We must choose $b < a$, so that choice of $\eta < \xi$ gives $\lambda_2 < 0$, which renders $b\lambda_2(\eta-\xi) < Y < a\lambda_2(\eta-\xi)$ as the interval over which equation (31) is defined. Without loss of generality, we take in this work $a = 2$ and $b = 1$. We consider the cases: $\eta = 1/3; \xi = 2/3$; $\eta = 1/4; \xi = 3/4$; $\eta = 0.49; \xi = 0.51$, to define layer thickness.

In expressions (15), in the absence of concrete data for the effective viscosities, we will choose $\vartheta_1 = \vartheta_2 = 1$ and $M_i = \frac{\mu_{ieff}}{\mu_i} = 1, i = 1,2,3$. The method of solution adopted in this work is still valid for different values of ϑ_1, ϑ_2 and M_i. Thus, $\lambda_1 = \frac{1}{\sqrt{2Da}}$; $\lambda_2 = \frac{1}{\sqrt[3]{2Da(\eta-\xi)}}$; $\lambda_3 = \frac{1}{\sqrt{Da}}$.

4.2. Choice of Darcy Number

Darcy number, Da, is the dimensionless permeability and has a maximum value of unity. We tested in this work the range of Da = 1; 0.1; 0.01; 0.001; 0.0001; 0.00001. It is clear that the lower the value of Da the higher the values of λ_1, λ_2 and λ_3. These values are given in **Table 2.** Equation (35) gives the following expressions for $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, which are tabulated in **Table 3**: $\gamma_1 = \lambda_2$; $\gamma_2 = \lambda_2(2\eta - \xi)$; $\gamma_3 = \frac{\pi}{\lambda_2^2}$ and $\gamma_4 = \frac{\pi}{\lambda_2}$.

Table 2. Values of λ_i, i=1,2,3 for choices of layer thicknesses and Darcy number

Da		Da=1	Da=0.1	Da=0.01	Da=0.001	Da=0.0001	Da=0.00001
$\eta = 1/3$, $\xi = 2/3$	λ_1	0.707107	2.2360689	7.071068	22.3606797	70.710678	223.606798
	λ_2	-1.144714	-2.4662121	-5.313293	-11.447142	-24.662121	- 53.132928
	λ_3	1	3.162278	10	31.622777	100	316.2277660
$\eta = 1/4$, $\xi = 3/4$	λ_1	0.7071067810	2.236067977	7.071067814	22.36067977	70.71067814	223.6067977
	λ_2	-1	-2.15443469	-4.64158883	-10	-21.544346	-46.4158883
	λ_3	1	3.16227766	10	31.62277660	100	316.227766
$\eta = 0.49$, $\xi = 0.51$	λ_1	0.707106781	2.236067977	7.07106781	22.360679	70.71067814	223.606797
	λ_2	-2.9240177	-6.2996052	-13.572088	-29.240177	-62.99605	-135.72088
	λ_3	1	3.1622777	10	31.62277660	100	316.227766

Table 3. Values of γ_j, j=1,2,3,4, for choices of layer thicknesses and Darcy number

Da		Da=1	Da=0.1	Da=0.01	Da=0.001	Da=0.0001	Da=0.00001
$\eta = 1/3$, $\xi = 2/3$	γ_1	-1.14471	-2.466212	-5.313293	-11.447142	-24.66212	-53.132928
	γ_2	0	0	0	0	0	0
	γ_3	2.397484	0.516522	0.111281	0.023975	0.0051652	0.001113
	γ_4	-2.744434	-1.273853	-0.591270	-0.274443	-0.127385	-0.059127
$\eta = 1/4$, $\xi = 3/4$	γ_1	-1	-2.154434	-4.6415888	-10	-21.54434	-46.415888
	γ_2	0.25	0.538608	1.1603972	2.5	5.386086	11.603972
	γ_3	3.141592	0.676835	0.14582	0.031416	0.0067683	0.001458
	γ_4	-3.14159	-1.458198	-0.676835	-0.314159	-0.14582	-0.067683
$\eta = 0.49$, $\xi = 0.51$	γ_1	-2.92401	-6.2996052	-13.572088	29.2401773	-62.99605	-135.72088
	γ_2	-1.37428	-2.960814	-6.378881	-13.742883	-29.60814	-63.788812
	γ_3	0.367443	0.0791632	0.017055189	0.003674429	0.000792	0.00017

	γ_4	-1.07441	-0.498696	-0.231475	-0.107441	-0.04987	-0.023147

4.3. Calculations of A_i, B_i and N_i using the selected parameters

In order to calculate the arbitrary constants appearing in the velocity profile and shear stress terms, shown in the next subsection, we need values of

$$A_i(\lambda_2[\eta-\xi]), B_i(\lambda_2[\eta-\xi]), N_i(\lambda_2[\eta-\xi]), A_i(2\lambda_2[\eta-\xi]), B_i(2\lambda_2[\eta-\xi]), N_i(2\lambda_2[\eta-\xi]),$$
$$A_i'(\lambda_2[\eta-\xi]), B_i'(\lambda_2[\eta-\xi]), N_i'(\lambda_2[\eta-\xi]), A_i'(2\lambda_2[\eta-\xi]), B_i'(2\lambda_2[\eta-\xi]), N_i'(2\lambda_2[\eta-\xi]).$$

Using the asymptotic approximations of **Table 1**, we approximate these values and produce **Tables 4(a)-4(d).** For small values of Da, the approximations become extremely large. Therefore, we restrict our attention to results using Da = 0.001 to Da = 1.

Table 4(a). Values of $A_i(\lambda_2[\eta-\xi]), B_i(\lambda_2[\eta-\xi]), N_i(\lambda_2[\eta-\xi])$ for choices of layer thicknesses and Darcy number

Da		Da=1	Da=0.1	Da=0.01	Da=0.001
$\eta=1/3$, $\xi=2/3$	A_i	0.259106	0.165765	0.04905	0.00138494
	B_i	0.792477	1.058740	2.511501	59.01173
	N_i	-0.023237	-0.110574	−0.6523860	−19.58325
$\eta=1/4$, $\xi=3/4$	A_i	0.231694	0.123404	0.0211112	0.00010834
	B_i	0.854277	1.28320	5.02709	657.792
	N_i	-0.04003	−0.196488	−1.52716	−219.199
$\eta=0.49$, $\xi=0.51$	A_i	0.339904	0.322532	0.285841	0.213048
	B_i	0.641164	−0.0005443	0.738865	0.902112
	N_i	-0.00054	−0.0025266	−0.0117383	−0.0549768

Table 4(b). Values of $A_i'(\lambda_2[\eta-\xi]), B_i'(\lambda_2[\eta-\xi]), N_i'(\lambda_2[\eta-\xi])$ for choices of layer thicknesses and Darcy number

Da		Da=1	Da=0.1	Da=0.01	Da=0.001
$\eta=1/3$, $\xi=2/3$	A_i'	−0.237682	−0.183409	−0.0710035	−0.00278963
	B_i'	0.501540	0.7488125	2.853765	110.9774
	N_i'	−0.122302	−0.2801354	−1.021887	−37.01812
$\eta=1/4$, $\xi=3/4$	A_i'	−0.224911	0.123404	−0.0341414	−0.000247414
	B_i'	0.544573	1.28320	6.94790	1435.82
	N_i'	−0.161651	−0.196488	−2.37563	−478.622

$\eta = 0.49$ $\xi = 0.51$	A_i'	-0.237682	-0.183409	-0.0710035	-0.00278963
	B_i'	0.501540	0.7488125	2.853765	110.9774
	N_i'	-0.122302	-0.2801354	-1.021887	-37.01812

Table 4(c). Values of $A_i(2\lambda_2[\eta-\xi]), B_i(2\lambda_2[\eta-\xi]), N_i(2\lambda_2[\eta-\xi])$ for choices of layer thicknesses and Darcy number

Da		Da=1	Da=0.1	Da=0.01	Da=0.001
$\eta = 1/3$ $\xi = 2/3$	A_i	0.176810	0.0587933	0.00238078	$1.33009\ 10^{-7}$
	B_i	1.01607	2.18260	35.6583	$4.33301\ 10^{5}$
	N_i	-0.094765	-0.533733	-11.7911	$-1.44434\ 10^{5}$
$\eta = 1/4$ $\xi = 3/4$	A_i	0.135292	0.0122423	0.000242211	$1.10475\ 10^{-10}$
	B_i	1.20742	0.0531997	305.491	$4.55641\ 10^{8}$
	N_i	-0.167255	0.0120870	-101.760	$-1.51880\ 10^{8}$
$\eta = 0.49$ $\xi = 0.51$	A_i	0.324847	0.290670	0.222157	0.110220
	B_i	0.667530	0.729679	0.878038	1.38479
	N_i	-0.0021774	-0.0101137	-0.0472823	-0.235643

Table 4(d). Values of $A_i'(2\lambda_2[\eta-\xi]), B_i'(2\lambda_2[\eta-\xi]), N_i'(2\lambda_2[\eta-\xi])$, for choices of layer thicknesses and Darcy number

Da		Da=1	Da=0.1	Da=0.01	Da=0.001
$\eta = 1/3$ $\xi = 2/3$	A_i'	-0.191405	-0.0826541	-0.00463577	$-3.71675\ 10^{-7}$
	B_i'	0.700355	2.34566	64.2671	$1.18234\ 10^{6}$
	N_i'	-0.256582	-0.853129	-21.4525	$-3.94113\ 10^{5}$
$\eta = 1/4$ $\xi = 3/4$	A_i'	-0.159147	-0.0431308	-0.000534159	$-3.52063\ 10^{-10}$
	B_i'	0.932436	5.27076	640.470	$1.42924\ 10^{9}$
	N_i'	-0.359249	-1.82079	-213.507	$-4.76413\ 10^{8}$
$\eta = 0.49$ $\xi = 0.51$	A_i'	-0.256529	-0.248917	-0.219838	-0.136680
	B_i'	0.452734	0.470224	0.563943	1.17072
	N_i'	-0.0372374	-0.0803695	-0.176277	-0.450232

4.4. Calculations of the arbitrary constants in the velocity profiles

Using the asymptotic approximations of **Table 1**, and the parameters and functions of sections 4.1-4.3 above, we can solve the matrix-vector equation (34) for the arbitrary constants appearing in the velocity profiles and the shear stress terms. Values of arbitrary constants for different layer thickness and various values of Darcy number are given in **Table 5**, below.

Table 5. Values of arbitrary constants, $a_1, a_2, b_1, b_2, c_1, c_2$, for choices of layer thicknesses and Darcy number

Da		Da=1	Da=0.1	Da=0.01	Da=0.001
$\eta = 1/3$ $\xi = 2/3$	a_1	−0.664631	−0.0239774	−0.000231142	$-6.23032\ 10^{-8}$
	a_2	−1.33537	−0.176023	−0.0197689	−0.00199994
	b_1	−0.807367	−0.291343	−0.0997572	−0.145990
	b_2	0.465189	0.161843	0.0371009	0.00799161
	c_1	−0.269959	−0.00415357	$-4.55516\ 10^{-7}$	$-1.84673\ 10^{-17}$
	c_2	−0.723537	−0.0442889	0.736007	38454.3
$\eta = 1/4$ $\xi = 3/4$	a_1	−0.664844	−0.0243588	−0.000310951	$-2.84904\ 10^{-7}$
	a_2	−1.33516	−0.175641	−0.0196890	−0.00199972
	b_1	−1.61511	−0.712949	−0.323800	−1.13782
	b_2	0.688040	0.216601	0.0486053	0.0104720
	c_1	−0.270011	−0.00416127	$-4.56230\ 10^{-7}$	$-1.84686\ 10^{-17}$
	c_2	−0.723152	−0.0399921	1.08239	$3.78771\ 10^{6}$
$\eta = 0.49$ $\xi = 0.51$	a_1	−0.664447	−0.0236392	−0.000170384	$-8.30069\ 10^{-9}$
	a_2	−1.33555	−0.176361	−0.0198296	−0.00199999
	b_1	0.139427	0.0796919	0.00956677	−0.000921628
	b_2	0.107055	0.0657693	0.0144172	0.00183344
	c_1	−0.269915	−0.00414758	$-4.55117\ 10^{-7}$	$-1.84673\ 10^{-17}$
	c_2	−0.723864	−0.0476304	0.542381	3065.34

4.5. Velocity and shear stress computations at the interfaces between layers

Velocity and shear stress term at the interface between the first and second layers take the following forms at $y = \eta$:

$$u_1(\eta) = a_1 \exp(\lambda_1 \eta) + a_2 \exp(-\lambda_1 \eta) + 2Da \qquad \ldots(49)$$

$$\frac{du_1}{dy}(\eta) = a_1 \lambda_1 \exp(\lambda_1 \eta) - a_2 \lambda_1 \exp(-\lambda_1 \eta). \qquad \ldots(50)$$

At the upper interface, $y = \xi$, velocity and shear stress term take the forms:

$$u_3(\xi) = c_1 \exp(\lambda_3 \xi) + c_2 \exp(-\lambda_3 \xi) + Da \qquad \ldots(51)$$

$$\frac{du_3(\xi)}{dy} = c_1 \lambda_3 \exp(\lambda_3 \xi) - \lambda_3 c_2 \exp(-\lambda_3 \xi). \qquad \ldots(52)$$

Numerical values of velocity and shear stress term at the interfaces are provided in **Table 6(a,b),** below, for different values of Darcy number and layer-thicknesses.

Table 6(a). Velocity and shear stress at the lower interface $y = \eta$.

Da		Da=1	Da=0.1	Da=0.01	Da=0.001
$\eta = 1/3$, $\xi = 2/3$	$u_1(\eta)$	0.103747	0.0659408	0.0156871	0.00189131
	$\frac{du_1}{dy}(\eta)$	−1.80251	−0.318990	−0.0272058	−0.00241784
$\eta = 1/4$, $\xi = 3/4$	$u_1(\eta)$	0.0877852	0.0569713	0.0148173	0.00191624
	$\frac{du_1}{dy}(\eta)$	−1.67983	−0.311624	−0.0284810	−0.00178064
$\eta = 0.49$, $\xi = 0.51$	$u_1(\eta)$	0.115829	0.0696752	0.0131867	0.00125557
	$\frac{du_1}{dy}(\eta)$	2.04900	0.189834	−0.0370620	−0.0166447

Table 6(b). Velocity and shear stress at the upper interface $y = \xi$.

Da		Da=1	Da=0.1	Da=0.01	Da=0.001
$\eta = 1/3$, $\xi = 2/3$	$u_3(\xi)$	0.102716	0.0604231	0.0105787	0.00102684
	$\frac{du_3}{dy}(\xi)$	−0.154332	−0.0911319	−0.0129460	−0.000850393
$\eta = 1/4$, $\xi = 3/4$	$u_3(\xi)$	0.0867926	0.0516770	0.00977377	0.00118938
	$\frac{du_3}{dy}(\xi)$	−0.230021	−0.129207	−0.0142354	−0.00601194
$\eta = 0.49$, $\xi = 0.51$	$u_3(\xi)$	0.115954	0.0703539	0.0139778	0.00157148
	$\frac{du_3}{dy}(\xi)$	0.00287020	−0.0297808	−0.0410000	−0.0180717

4.6. Mean velocity across the layers

Equation (50) provides an expression for the mean velocity across each layer and across the channel composed of the three layers. For different layer thicknesses and various Darcy number values, **Table 7** provides numerical values for the mean velocity.

Table 7. Mean velocity across the composite layers for different layer thicknesses and various values of Da.

		Da=1	Da=0.1	Da=0.01	Da=0.001
$\eta = 1/3$, $\xi = 2/3$	$\bar{u}$	0.0775308	0.0480058	0.0107074	0.000875
$\eta = 1/4$, $\xi = 3/4$	$\bar{u}$	0.0775269	0.0479474	0.004101	0.000632
$\eta = 0.49$, $\xi = 0.51$	$\bar{u}$	0.0397180	0.0236304	0.00450508	0.00136549

4.7. Permeability distribution across the layers

We have assumed that layer 1 and layer 3 are of constant permeability, and layer 2 of variable permeability. The dimensionless permeability in layer 1 is given by $K_1 = aK_0$, and that in layer 3 by $K_3 = bK_0$, where we have chosen $b = 1, a = 2$. Dimensionless permeability in layer 2 is given by $K_2 = \frac{2K_0(\eta - \xi)}{-y + 2\eta - \xi}$. The above permeability distributions, K_1, K_2 and K_3 can be rewritten to replace K_0 by Da as:

$$K_1^* = \frac{K_1}{H^2} = 2Da \quad ...(53)$$

$$K_2^* = \frac{K_2}{H^2} = \frac{2Da(\eta - \xi)}{-y + 2\eta - \xi} \quad ...(54)$$

$$K_3^* = \frac{K_3}{H^2} = Da\ . \quad ...(55)$$

Permeability distributions in (56) is tabulated in **Table 7**, below, for each thickness. Permeability distributions across the layers are plotted for the selected range of Da and layer thicknesses in **Figs 2** to **4**. These figures show the constant permeability in the lower and upper bounding layers, and the variable permeability (decreasing) in the middle layer.

Table 7. Variable Permeability function for each middle layer thickness

	$K^*_2 = \frac{2Da(\eta - \xi)}{-y + 2\eta - \xi}$
$\eta = 1/3$ $\xi = 2/3$	$\frac{2Da}{3y}$
$\eta = 1/4$ $\xi = 3/4$	$\frac{Da}{y + 1/4}$
$\eta = 0.49$ $\xi = 0.51$	$\frac{4Da}{100y - 47}$

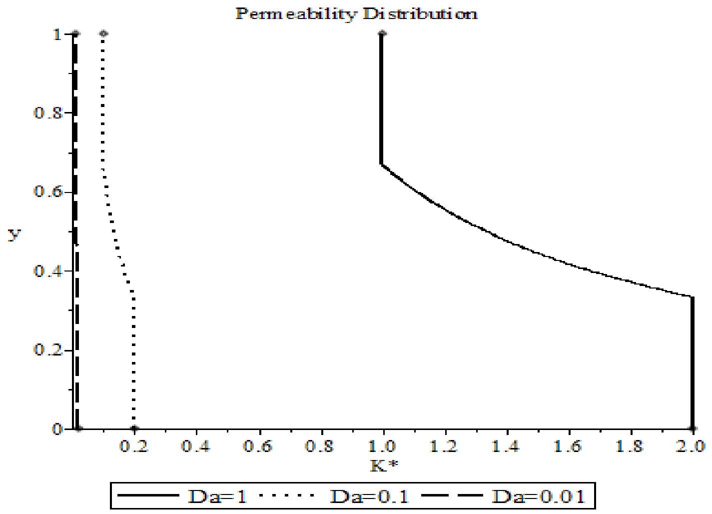

Figure 2. Permeability Distribution $\eta = \frac{1}{3}, \xi = \frac{2}{3}$, and different values of Darcy number.

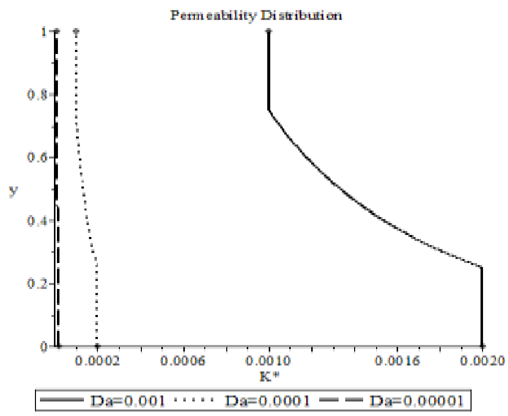

Figure 3. Permeability Distribution $\eta = \frac{1}{4}, \xi = \frac{3}{4}$, **and different values of Darcy number.**

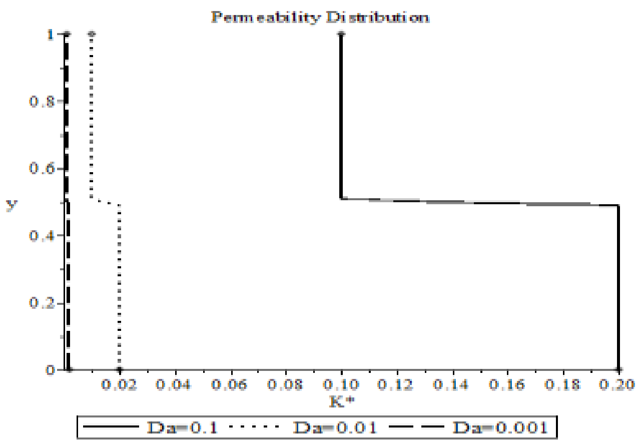

Figure 4. Permeability Distribution $\eta = \frac{1}{4}, \xi = \frac{3}{4}$, **and different values of Darcy number.**

4.8. Velocity distribution across the layers

Velocity distributions across the variable permeability middle layer, for different layer thicknesses and Darcy number, are illustrated in **Figs 5** through **7**. These figures show the shape and demonstrate the increase in velocity with increasing Da. Velocity distributions across the composite three layers, for different layer thicknesses and Darcy number, are illustrated in **Figs 8** through **10**. These figures show the shape, smoothness in the velocity profiles, and demonstrate the increase in velocity with increasing Da.

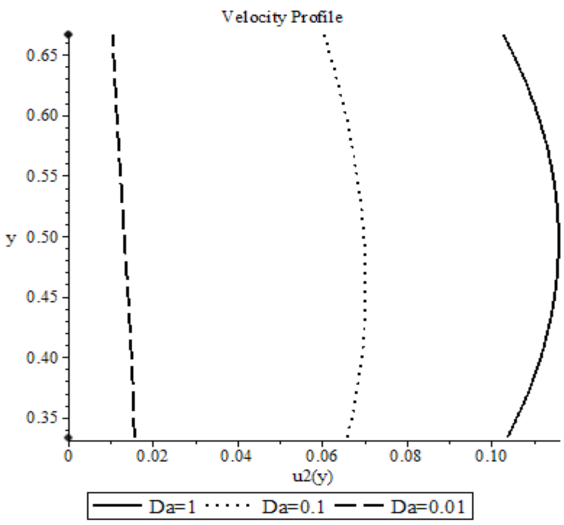

Figure 5. Velocity profiles in the second layer for $\eta = \frac{1}{3}, \xi = \frac{2}{3}$, **and different values of Darcy number.**

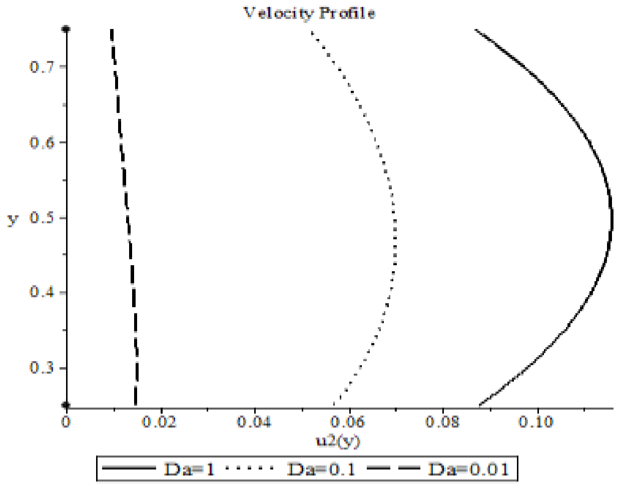

Figure 6. Velocity profiles in the second layer for $\eta = \dfrac{1}{4}, \xi = \dfrac{3}{4}$**, and different values of Darcy number.**

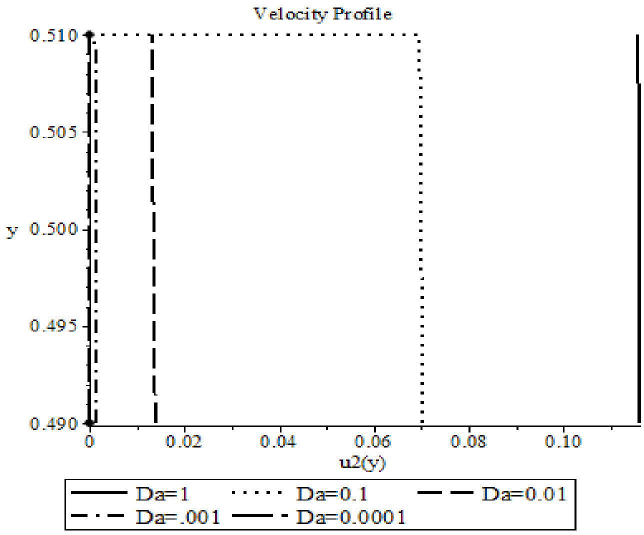

Figure 7. Velocity profiles in the second layer for $\eta = 0{,}49, \xi = 0.51$**, and different values of Darcy number.**

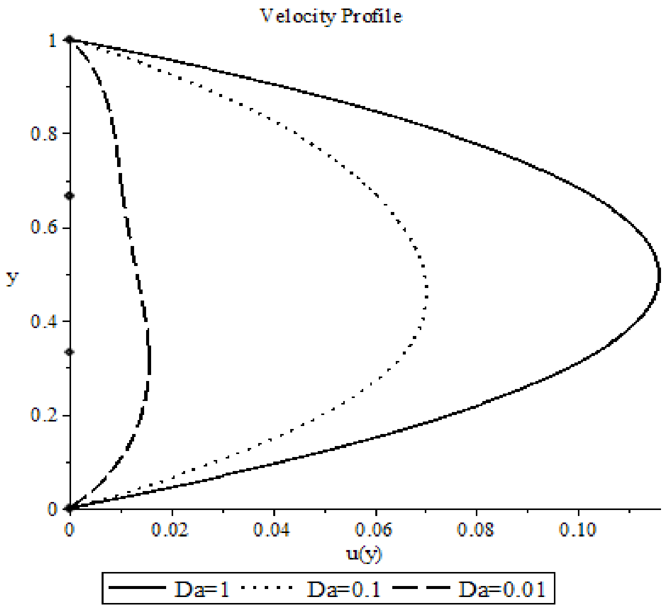

Figure 8. Velocity profiles in the three layers for $\eta = \dfrac{1}{3}, \xi = \dfrac{2}{3}$**, and different values of Darcy number.**

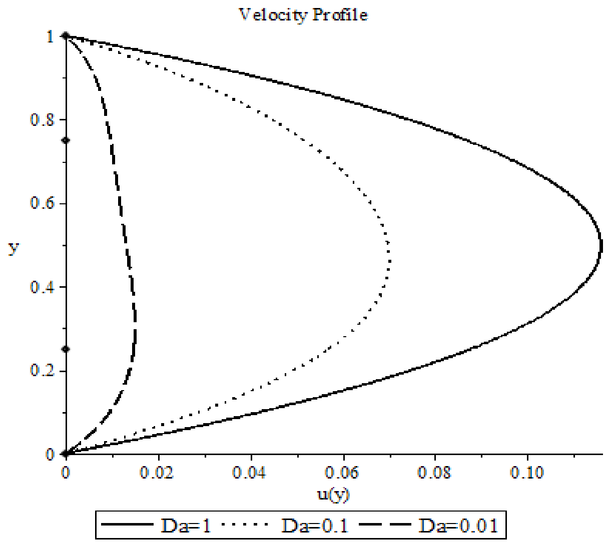

Figure 9. Velocity profiles in the three layers for $\eta = 0.25, \xi = 0.75$, and different values of Darcy number.

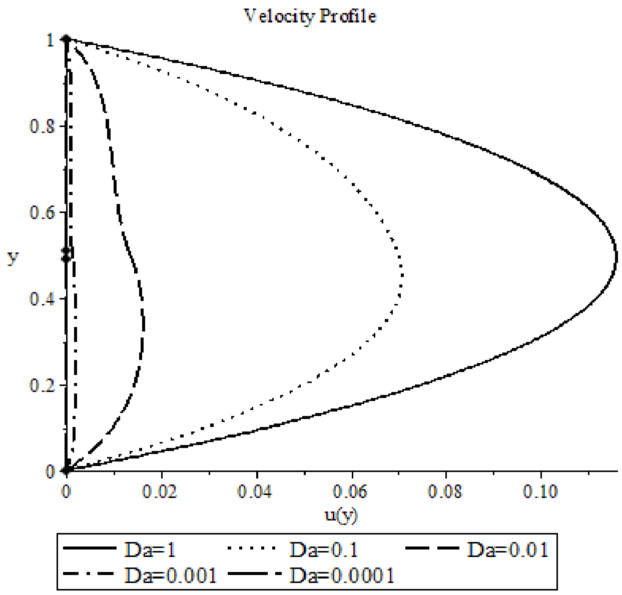

Figure 10. Velocity profiles in the three layers for $\eta = 0.49, \xi = 0.51$, and different values of Darcy number.

CONCLUSION

In this work we considered the flow through a composite of layered media, composed of three layers the flow through which is governed by Brinkman's equation. The middle layer is taken to be of variable permeability and the bounding layers of constant permeability. Flow through the middle layer is governed by an Airy's differential equation. Its solution has been obtained and expressed in terms of the Nield-Kuznetsov function, and characteristics of the flow in the configuration have been thoroughly analyzed.

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