On Nill-Pure Rings

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ABSTRACT

Let R be a ring. The ring R is called right Nil-pure, if for any $a \in R$, r(a) is an aleft pure ideal of R . In this paper, we give some characterizations and properties of right Nil-pure rings, which is a proper generalization of every ideal of R is pure. And we study the regularity of right Nil-pure ring. For example:

1- Let R be a reversible and Nil-pure ring. Then R is n-regular ring
2- Let R be ZI-ring with every simple singular right R-module is almost nil –injective, then R is nil-pure ring

Keyword: Nill Pure Rings, Ring, Pure Ideal, right annihilater

1. INTRODUCTION

Throughout this paper, R will be associative ring with identity and M is aright R-module with S=End (M_R). The center of a ring , the set of all nilpotent elements in R , the right singular of R and the Jacobson radical of R are denoted by $C(R)$, $N(R), Y(R)$, and $J(R)$ respectively . We write for any $a \in R$, $r(a)$ and $l(a)$ the right annihilater of a and the left annihilater of a respectively .

Standard references like [1] , [2] , and [6] have motivated many authors for study pure ideals . An ideal I of a ring R is called right pure if for every $a \in I$ there exists $b \in I$ such that$a = ab$. It is known that a ring R is $PF-ring$ if and only if for every $a \in R$, $ann(a)$ is pure (R is comm-ring). See Al-Ezeh [1]

A cording to Cohn [3] a ring R is called reversible if $ab = 0$ implies that $ba = 0$ for $a, b \in R$. A ring R is called a $ZI-ring$ if fora, $b \in R$, $ab = 0$ implies $aRb = 0$. Every reversible ring is $ZI-ring$ and every idempotent is central. A ring R is called regular [5] , if for every $a \in R$ there exisits $b \in R$ such that $a = aba$. A ring R is called $n-regular$ if $a \in aRa$ for all $a \in N(R)$ [7]. clearly regular rings are $n-regular$ rings , but the converse is not true by [7] . A ring R is called reduced if R contains no non-zero nilpotent elements . or equivalently , $a^2 = 0$ implies $a = 0$ in R for all $a \in R$. Clearly , areduced ring is $n-regular$

In this paper , we introduced the definition of right $Nil-Pure$ rings , giving some characterizations and properties . Some important results which are known for every ideal of R is pure are hold for right Nil-pure rings and we study the relation between of$n-regular$ rings and reduced rings in term of them.

2. CHARACTRRIZATION OF RIGHT NIL-PURE RINGS

Definition. 2.1: A ring R is called right**Nil-pure** ring, if for each $a \in N(R)$, r(a) is a left pure ideal

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Example:
1- Let Z_{12} be the ring of integers modulo 12 .

$\mathrm{r}(6) = \{0, 2, 4, 6, 8, 10\}$ not pure ideal .
there fore Z_{12} is not Nil - Pure ring .
2 – Let R be the ring of 2×2 matrices

Where Z_2 is the ring of integers modulo 2.

Clearly $r\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right)$Is a left pure ideal.
Then R is right Nil-pure ring

3- The ring Z of integers isNil-pure ring

Theorem:-2.2

Let R be rightNil-pure ring and I is pure ideal of R . Then R/I is Nil-Pure ring

Proof:

Assume that a$+I \in N(R/I)$. Since R isNil-Purering Thenr (a) is a left pure , $a \in N(R)$.
We shall show that $r(a+I)$ is aleft pure in R/I .
Let $x+I \in r(a+I)$, then $ax \in I$. Since I is right pure ideal , then there exists $y \in I$ such that $ax = (ax)y$
Implies that $a(x-xy) = 0$ andx$-xy \in r(a)$. Since $r(a)$ is a left pure , then
$(x-xy) = z(x-xy)$, for some $z \in r(a)$

$$x - xy = zx - zxy$$
$$x - zx = xy - zxy$$
$$= (x - zx)y \in I$$

Therefore $x - zx \in I$.
Hence $x + I = (z+I)(x+I)$
So $r(a+I)$ is a left pure . Consequently R/I is Nil-pure ring .■

Lemma: 2.3. [3]

Let R be a ZI-ring . Then r(a) = l(a) , for every $a \in R$.

Let R be a ZI-ring . Then R is right Nil-Pure ring if and only if $r(a) + r(b) = R$ for some $a, b \in N(R)$ with $ab = 0$

Proof:

Suppose that Ris Nil-Pure ring , then $r(a)$ is a left pure $(a \in N(R))$. Assume that r(a)+r(b)≠R . Then there exists a maximal right ideal M of R containing r(a)+r(b) . Since $ab = 0$ then $b \in r(a)$ and $b = cb$ for some c∈r(a) .
Hence $(1 - ca) \in l(b) = r(b) \subseteq M$ $[Lemma\ (2.2)]$ but $c \in r(a) \subseteq M$, yielding $1 \in M$, which is a contradiction $M \neq R$.
Therefore $r(a) + r(b) = R$

Conversly: Assume that $r(a) + r(b) = R$ for all $a, b \in N(R)$ with $ab = 0$
In particular $c + d = 1$, $c \in r(a)$, $d \in r(b) = l(b)$
Hence $cb + db = b$. therefore $cb = b$
So R is $Nil - pure$ ring.■

3. REGULARITY OF RIGHT NIL-PURE RINGS

Proposition: 3.1.

Let R be aright Nil-Pure ring .Then $C(R)$ contains no non zero nilpotent element .

Proof:

Let $x \in C(R)$ with $x^n = 0$ and $x^{n-1} \neq 0$ for some $n \in Z^+$
Since R is $Nil - Pure$ ring , then $r(x)$ is a left pure and $x^{n-1} \in r(x)$, $yx^{n-1} = x^{n-1}$for some $y \in r(x)$. Since $xy = 0$ and $x \in C(R)$, $x = 0$. So $x^{n-1} = (yx)x^{n-2} = 0$ is a contradiction .Therefore $C(R)$ contains no nonzero nilpotent element .■

Corollary: 3.2.

Let R be Nil-Pure ring . Then C(R) is reduced.

Theorem: 3.3

Let R be a reversible and Nil-Pure ring . Then R is a reduced ring.

Proof:

let $a \in R$ such that $a^2 = 0$, since R is Nil-Purering then $r(a)$ a left pure and $xa = a$ for some $x \in r(a)$. since R is reversible and $ax = 0$, then $xa = 0$. so $a = xa = 0$. Hence R is reduced.■

Lemma: 3.4 [9]
If $Y(R)$ resp $(Z(R))$ is reduced, then $Y(R) = 0$ $(resp(Z(R))$.
By Theorem [3.3] and Lemma [3.4] , we can easily obtain the following corollary:

Corollary: 3.5.

If R is a reversible, right Nil-Pure ring , Then R is a left and right non-singular .

Remark:
Everyn-regular ring is Nil-pure ring but the converse is not true, [Example 2]
Explain: from example clean that Nil-Pure ring but not n-regular because that

The following result contains a sufficient condition $for\ n - regulr$ rings interms of nil-pure rings.

Theorem: 3.6.

Let R be a reversible ring. Then R is Nil-Pure ring if and only ifR is n-regular.
Proof:
Assume that R isNil-Purering, then by [3.3] R is
n-regular.

Conversely: It is clear

In[10] , Zhao and Du first introduced and characterized aright almost nill-injectivering , and gave many properties . Let M_R be a module withS= $End(M_R)$. The module M is called right nil-injective, if for any $k \in N(R)$, there exists an S-submodule X_k of M such that $l_M r_R(k) = Mk \oplus X_k$ as left S-module .If R_R isalmost nil-injective, then we call R a right **almost nil-injective ring**

Lemma: 3.7 [10]
Suppose M is a right R-module with S=End(M_R) . If $l_M r_R(a) = Ma \oplus X_a$, Where X_a is a left S-submodule of M_R . set $f: aR \rightarrow$M a right R-homomorphism , then f(a)=ma+x with $m \in M$, $x \in X_a$.

Theorem:3.8

Let R be aZI-ring with every simple singular rightR-module is right almost nil-injective. Then R is right nil-pure ring .
Proof:
Let $0 \neq a \in N(R)$ such that $ab = 0$ for some b∈N(R). Suppose that r(a)+r(b)≠R , then there exists a maximal right ideal M of R containing $r(a) + r(b)$. If M is not essential in R thenM= $r(e)$, $e^2 = e \in R$. But $ab = 0$ gives b∈l(b)=r(b)⊆M⊆r(e) , where it follows that $ea = 0$, yielding $e \in l(a) = r(a) \subseteq M \subseteq r(e)$, which is a contradiction .Hence M is essential in R . Thus R/M is almost nil-injective and $l_{R/M} r_R(a) = (R/M)a \oplus X_a$, $X_a \leq R/M$. Let $f: aR \longrightarrow R/M$ be defiend by $f(ar) = r + M$. Note that f is a well-defined R-homomorphism. Then by [Lemma 3.7] $1 + M = f(a) = ca + M + x$, $c \in R$, $x \in X_a$, $1 - ca + M = x \in R/M \cap X_a = 0$, $1 - ca \in M$, Since R is ZI-ring , $ca \in r(a)$, then $1 \in M$, which is a contradiction. Therefore $r(a) + r(b) = R,$ where $ab = 0$. Thus R is Nil-Pure ring (Proposition 2.4) .■

According to Lwei , wang and Li, [8] , a right ideal L of R is called an N-ideal , if for every $b \in N(R) \cap L$, $bR \subseteq L$, A ring R is called NZI if for any $a \in R$, $r(a)$ is an N-idealof R .Clearly, ZI- rings are NZI , but the converse is not true , in general [8,Example 2.1]

Proposition: 3.9.

The following conditions are equivalent for any reversible ring R .

1- R is reduced
2- R is Nil-Pure ring
3- R is an n-regular ring and ZI- ring
4- R is an n-regular ring and NZI- ring

5- R is NZI- ring and Every simple right R-module is almost nil-injective ring.

Proof:

$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$ is trivial

$5 \rightarrow 1$: Let $a \in R$ and $a^2 = 0$. Ifa$\neq 0$, then $r(a) \neq R$, so there exist a maximal right ideal M of R such that $r(a) \subseteq M$
Since R/M is simple rigt R-module , R/M is almot nil-injective and $l_{R/M}r_R(a) = (R/M)\oplus X_a$, $X_a \subseteq R/M$
Let $f: aR \rightarrow R/M$ be defiend by f(ar)=r+M . Note that f is a well defiend R-homomorphism thus $1 + M = f(a) = ca + M + x$,c$\in R$,$x \in X_a$ [Lemma 3.7]
Hence $1 - ca + M = x \in R/M \cap X_a = 0$, $1 - ca \in M$
Since R is NZI- ring $ca \in r(a)$, then $1 \in M$,which is a contradiction . Therefore $a = 0$ and R is reduced .■

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