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Well-balanced Alternative WENO Scheme for Ten-moment Gaussian Closure Equations

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ABSTRACT

In this work, we consider the Ten-Moment equations with source terms, which arise in various applications related to plasma flows. We develop a well-balanced, high-order conservative finite difference scheme based on the Alternative Weighted Essentially Non-Oscillatory (AWENO) method. The proposed scheme accommodates arbitrary consistent Riemann solvers, such as Lax-Friedrichs (LF), HLLC, and others, within the AWENO framework. To preserve hydrostatic equilibrium, we incorporate a hydrostatic reconstruction technique. Furthermore, we introduce a slight modification to the original hydrostatic reconstruction approach that avoids spurious oscillations and enables fifth-order accuracy within the finite difference context. Importantly, in the absence of source terms, our discretization remains fully conservative—a property not generally retained by methods employing standard hydrostatic reconstruction. We also demonstrate that our approach ensures the well-balanced property at the discrete level, providing an accurate and robust numerical treatment of the Ten-Moment system under equilibrium conditions.

Keywords: Finite differece method, Well-balanced scheme, AWENO scheme, Ten-Moment equations.

INTRODUCTION

A fluid flow model based on the Ten-Moment equations [3] is derived by taking higher moments of the Boltzmann equation, capturing anisotropic effects that are typically neglected when assuming local thermodynamic equilibrium, as in the derivation of the Euler equations for compressible flow [10, 11]. This leads to a system of equations that includes the symmetric pressure tensor, allowing for more accurate modeling in scenarios where anisotropy is important. The Ten-Moment model performs well in a range of situations, but it is particularly effective in plasma flow applications, where capturing the anisotropic behavior of the plasma is essential.

To solve such complex mathematical models, we use numerical schemes (also referred to as numerical methods or numerical algorithms). These methods discretize the problem domain, replacing continuous equations with a system of discrete equations that can be solved computationally. Numerical methods are widely used in research and engineering when analytical solutions are difficult or impossible to obtain. By breaking down the problem into simpler spatial elements or time steps, these methods yield approximate solutions to complex systems.

A variety of numerical methods are employed to approximate solutions of hyperbolic partial differential equations (PDEs) [7, 8, 15], such as advection equations, Burger's equations and various systems of conservation laws. In recent years, high-order methods have played a crucial role in accurately replicating solutions to hyperbolic conservation laws. These methods can produce numerical solutions that closely match the classical (analytical) solutions in regions with smooth flow. However, the presence of discontinuities introduces challenges, as standard methods may produce spurious oscillations near such features. To mitigate this, non-oscillatory schemes are essential.

Among the effective approaches are the ENO (Essentially Non-Oscillatory) and WENO (Weighted ENO) schemes. The WENO scheme was first introduced in [23], sparking extensive research aimed at enhancing resolution, reducing numerical diffusion, and accurately capturing discontinuities without introducing oscillations. A notable advancement was made by Borges et al. in [4], who proposed improved nonlinear weights to transition from the classical WENO-JS scheme to the more accurate WENO-Z scheme. Further developments include characteristic-wise weighted compact nonlinear (WCN) schemes introduced in [5], which are based on WENO interpolation of point values. Similarly, the conservative finite difference AWENO schemes presented in [16] share foundational principles with WCN schemes.

High-order well-balanced methods have also become a significant area of recent research. For instance, a well-balanced finite difference WENO scheme tailored for isothermal (constant temperature) equilibrium was proposed in [17]. In that work, well-balancing was achieved by linearizing specific WENO operators. Additionally, researchers in [16] applied well-balanced discontinuous Galerkin (DG) methods and finite volume WENO schemes to hyperbolic systems with source terms. Their approach relied on the hydrostatic reconstruction technique, initially introduced in [1, 2].

The primary goal of this work is to develop a finite difference AWENO scheme combined with hydrostatic reconstruction for the ten-moment equations. To compute interpolated values at cell interfaces, we adopt the approach from [17], reformulating the source term and employing scale-invariant (Si) WENO interpolation [10] with characteristic projections. To ensure that the proposed scheme is well-balanced, the Si-WENO interpolated values are further modified using the hydrostatic reconstruction technique introduced in [17]. A key focus of this work is the observation that fifth-order accuracy cannot be achieved directly when applying hydrostatic reconstruction. In [16], the one-sided numerical flux was implemented within finite volume and discontinuous Galerkin (DG) frameworks, with the hydrostatic reconstruction strategy specifically developed for the shallow water equations. We reconstruct a high-order conservative finite difference AWENO scheme for general systems of hyperbolic conservation laws. Further, we apply this approach to the ten-moment equations. Additionally, we introduce a hydrostatic reconstruction strategy tailored for the finite difference method and demonstrate that the resulting scheme is well-balanced.

GOVERNING MODEL

In two-dimensions, the system of ten-moment equations is given by

$$\partial_t U + \partial_x G^x(U) + \partial_y G^y(U) = S(U) \tag{1}$$

where,

$$U = \begin{bmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ E_{11} \\ E_{12} \\ E_{22} \end{bmatrix}$$

is the vector of conservative variables. and,

$$G^x(U) = \begin{bmatrix} \rho v_1 \\ \rho (v_1)^2 + p_{11} \\ \rho v_1 v_2 + p_{12} \\ (E_{11} + p_{11}) v_1 \\ E_{12} v_1 + \frac{1}{2}(p_{11} v_2 + p_{12} v_1) \\ E_{22} v_1 + p_{12} v_2 \end{bmatrix}, \qquad G^y(U) = \begin{bmatrix} \rho v_2 \\ \rho v_1 v_2 + p_{12} \\ \rho (v_2)^2 + p_{22} \\ E_{11} v_2 + p_{12} v_1 \\ E_{12} v_2 + \frac{1}{2}(p_{12} v_2 + p_{22} v_1) \\ (E_{22} + p_{22}) v_2 \end{bmatrix} \tag{2}$$

are flux components. And the source terms $S(U) = S^x(U) + S^y(U)$ are given by

$$S^x(U) = \begin{bmatrix} 0 \\ -\frac{1}{2}\rho \partial_x W \\ 0 \\ -\frac{1}{2}\rho v_1 \partial_x W \\ -\frac{1}{4}\rho v_2 \partial_x W \\ 0 \end{bmatrix}, \qquad S^y(U) = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}\rho \partial_y W \\ 0 \\ -\frac{1}{4}\rho v_1 \partial_y W \\ -\frac{1}{2}\rho v_2 \partial_y W \end{bmatrix} \tag{3}$$

Here, ρ is density, v_1 and v_2 are components of velocity **v** and E_{11},E_{12} and E_{22} are components of symmetric energy tensor

$$E = [E_{ij}]_{\{i,j\} \in \{1,2\}}$$

The source term is depending upon ∂W for a given function W(x,y,t), i.e. based on external force acting on flow. Here,the ideal gas equation of state (EOS) is considered to close the system i.e.:

$$E_{ij} = \frac{1}{2}(\rho v_i v_j + p_{ij}), \qquad \forall \{i,j\} \in \{1,2\}$$

where, p_{11},p_{12} and p_{22} are components of symmetric pressure tensor

$$P = [p_{ij}]_{\{i,j\} \in \{1,2\}}.$$

In one-dimension, the system of ten-moment equation has following form:

$$\partial_t U + \partial_x G^x(U) = S^x(U)$$

and , the above system without source term will be hyperbolic as jacobian matrix of flux function $G^x(U)$ has distinct eigenvalues and linearly independent eigenvectors as follows:

$$\lambda_1 = v_1 - \sqrt{3p_{11}/\rho}, \quad \lambda_2 = v_1 - \sqrt{p_{11}/\rho}, \quad \lambda_3 = v_1, \lambda_4 = v_1, \quad \lambda_5 = v_1 + \sqrt{p_{11}/\rho}, \quad \lambda_6 = v_1 + \sqrt{3p_{11}/\rho}$$

and corresponding left and right eigenvectors are given as

$$L = \begin{bmatrix} \frac{v_1}{2cp_{11}\rho} + \frac{v_1^2}{6p_{11}^2} & -\frac{1}{2cp_{11}\rho} - \frac{v_1}{3p_{11}^2} & 0 & \frac{1}{3p_{11}^2} & 0 & 0 \\ \frac{\sqrt{3}v_2}{2c\rho} - \frac{\sqrt{3}p_{12}v_1}{2c\rho p_{11}} + \frac{v_1v_2}{2p_{11}} - \frac{p_{12}v_1^2}{2p_{11}^2} & \frac{\sqrt{3}p_{12}}{2c\rho p_{11}} - \frac{v_2}{2p_{11}} + \frac{p_{12}v_1}{p_{11}^2} & -\frac{\sqrt{3}}{2c\rho} - \frac{v_1}{2p_{11}} & -\frac{p_{12}}{p_{11}^2} & \frac{1}{p_{11}} & 0 \\ 1 - \frac{\rho v_1^2}{3p_{11}} & \frac{2\rho v_1}{3p_{11}} & 0 & -\frac{2\rho}{3p_{11}} & 0 & 0 \\ v_2^2 - \frac{2p_{12}v_1v_2}{p_{11}} + \frac{4p_{12}^2v_1^2}{3p_{11}^2} - \frac{p_{22}v_1^2}{3p_{11}} & \frac{2p_{12}v_2}{p_{11}} - \frac{8p_{12}^2v_1}{3p_{11}^2} + \frac{p_{22}v_1}{3p_{11}} & -2v_2 + \frac{2p_{12}v_1}{p_{11}} & \frac{8p_{12}^2}{3p_{11}^2} - \frac{2p_{22}}{3p_{11}} & -\frac{4p_{12}}{p_{11}} & 2 \\ -\frac{\sqrt{3}v_2}{2c\rho} + \frac{\sqrt{3}p_{12}v_1}{2c\rho p_{11}} + \frac{v_1v_2}{2p_{11}} - \frac{p_{12}v_1^2}{2p_{11}^2} & -\frac{\sqrt{3}p_{12}}{2c\rho p_{11}} - \frac{v_2}{2p_{11}} + \frac{p_{12}v_1}{p_{11}^2} & \frac{\sqrt{3}}{2c\rho} - \frac{v_1}{2p_{11}} & -\frac{p_{12}}{p_{11}^2} & \frac{1}{p_{11}} & 0 \\ -\frac{v_1}{2cp_{11}\rho} + \frac{v_1^2}{6p_{11}^2} & \frac{1}{2cp_{11}\rho} - \frac{v_1}{3p_{11}^2} & 0 & \frac{1}{3p_{11}^2} & 0 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} \rho p_{11} & 0 & 1 & 0 & 0 & \rho p_{11} \\ \rho v_1 p_{11} - c\rho p_{11} & 0 & v_1 & 0 & 0 & \rho v_1 p_{11} + c\rho p_{11} \\ \rho v_2 p_{11} - c\rho p_{12} & -\frac{c\rho}{\sqrt{3}} & v_2 & 0 & \frac{c\rho}{\sqrt{3}} & \rho v_2 p_{11} + c\rho p_{12} \\ \frac{3p_{11}^2}{2} - c\rho v_1 p_{11} + \frac{\rho v_1^2 p_{11}}{2} & 0 & \frac{v_1^2}{2} & 0 & 0 & \frac{3p_{11}^2}{2} + c\rho v_1 p_{11} + \frac{\rho v_1^2 p_{11}}{2} \\ \frac{3p_{11}p_{12}}{2} - \frac{c\rho v_1 p_{12}}{2} - \frac{c\rho v_2 p_{11}}{2} + \frac{\rho v_1 v_2 p_{11}}{2} & \frac{p_{11}}{2} - \frac{c\rho v_1}{2\sqrt{3}} & \frac{v_1 v_2}{2} & 0 & \frac{p_{11}}{2} + \frac{c\rho v_1}{2\sqrt{3}} & \frac{3p_{11}p_{12}}{2} + \frac{c\rho v_1 p_{12}}{2} + \frac{c\rho v_2 p_{11}}{2} + \frac{\rho v_1 v_2 p_{11}}{2} \\ \frac{p_{11}p_{22}}{2} + p_{12}^2 - c\rho v_2 p_{12} + \frac{\rho v_2^2 p_{11}}{2} & p_{12} - \frac{c\rho v_2}{\sqrt{3}} & \frac{v_2^2}{2} & \frac{1}{2} & p_{12} + \frac{c\rho v_2}{\sqrt{3}} & \frac{p_{11}p_{22}}{2} + p_{12}^2 + c\rho v_2 p_{12} + \frac{\rho v_2^2 p_{11}}{2} \end{bmatrix}$$

where, $c = \sqrt{3p_{11}/\rho}$ is speed of sound.

CONSERVATIVE FINITE DIFFERENCE AWENO SCHEME

In this section, we discuss the general conservative finite difference AWENO scheme. For simplicity, we omit the source term and consider the one-dimensional hyperbolic conservation law:

$$\frac{\partial U}{\partial t} + \frac{\partial G^x(U)}{\partial x} = 0 \tag{4}$$

with suitable initial and boundary conditions. System with source term brought by a function W(x,y,t) which is based on external force acting on flow will be discussed in further section.

Let us assume that domain is uniformly partitioned i.e.,

$$a = x_{1/2} < x_{3/2} < ... < x_{N-1/2} < x_{N+1/2} = b$$

where $x_{i+1/2}$ represents cell interfaces, with cell centers as:

$$x_i = x_{i-1/2} + \frac{\Delta x}{2}$$

where $\Delta x = \frac{b-a}{N}$ represents measure of cell. For simplicity, we consider the following one-dimensional scalar equation as:

$$\frac{\partial u}{\partial t} + \frac{\partial g(u)}{\partial x} = 0 \tag{5}$$

A semi-discretized form of (5) in space is expressed as:

$$\frac{\partial u}{\partial t} + \frac{\hat{g}_{i+1/2} - \hat{g}_{i-1/2}}{\Delta x} = 0 \tag{6}$$

where $\hat{g}_{i\pm 1/2}$ represents numerical fluxes which will be discussed in further sub-sections.

WENO INTERPOLATION PROCEDURE

Here, we consider the fifth order WENO interpolation procedure. Details for this can be seen in [4, 9]. When discontinuities are observed in stencil, then values of function can be approximated to the left or right of cell interfaces $x_{i+1/2}$ which is denoted by $u^-_{i+1/2}$ and $u^+_{i+1/2}$ respectively. Here, WENO interpolation procedure is discussed only for $u^-_{i+1/2}$ because interpolation procedure for $u^+_{i+1/2}$ can be obtained by mirror-symmetric way. The interpolated value $x_{i+1/2}$ is obtained by using:

$$u^-_{i+1/2} = W^-_{i+1/2}[u] = \sum_{k=0}^{2} w_k u^{(k)}(x_{i+1/2}) \tag{7}$$

where $W^-_{i+1/2}[.]$ represents the fifth order WENO interpolation operator and $u^{(k)}$ are calculated by using second degree Legrange interpolation polynomials:

$$\begin{aligned}
u^{(0)}(x_{i+1/2}) &= \frac{3}{8}u_{i-2} - \frac{5}{4}u_{i-1} + \frac{15}{8}u_i, \\
u^{(1)}(x_{i+1/2}) &= -\frac{1}{8}u_{i-1} + \frac{3}{4}u_i + \frac{3}{8}u_{i+1}, \\
u^{(2)}(x_{i+1/2}) &= \frac{3}{8}u_i + \frac{3}{4}u_{i+1} - \frac{1}{8}u_{i+2}
\end{aligned} \tag{8}$$

The nonlinear weights w_k,k=0,1,2 in [4] are defined as:

$$\begin{aligned}
w_k &= \frac{\lambda_k}{\Sigma_{s=0}^{2}\lambda_s} \\
\lambda_k &= d_k\left(1 + \left(\frac{\mu_5}{\sigma_k + \xi}\right)^p\right)
\end{aligned} \tag{9}$$

where,

$$d_0 = \frac{1}{16}, \quad d_1 = \frac{5}{8}, \quad d_2 = \frac{5}{16}$$

and parameter ξ is used to avoid the denominator to be zero.
where,

$$\mu_5 = |\sigma_0 - \sigma_2|$$

Here, value of ξ and p is used as

$$\xi = 10^{-12} \quad and \qquad p = 2$$

Here μ_5 is the global optimal order smoothness indicator with local smoothness indicators σ_k which is computed by:

$$\begin{aligned}
\sigma_0 &= \frac{13}{12}(u_{i-2} - 2u_{i-1} + u_i)^2 + \frac{1}{4}(u_{i-2} - 4u_{i-1} + 3u_i)^2 \\
\sigma_1 &= \frac{13}{12}(u_{i-1} - 2u_i + u_{i+1})^2 + \frac{1}{4}(u_{i-1} - u_{i+1})^2 \\
\sigma_2 &= \frac{13}{12}(u_i - 2u_{i+1} + u_{i+2})^2 + \frac{1}{4}(3u_i - 4u_{i+1} + u_{i+2})^2
\end{aligned} \tag{10}$$

0.1 HIGH-ORDER RECONSTRUCTION

In case of finite difference methods, a fifth order discretization is given by

$$\frac{1}{\Delta x}(\hat{g}_{i+1/2} - \hat{g}_{i-1/2}) = \frac{dg(u)}{dx}|_{x_i} + O(\Delta x^5), \tag{11}$$

where $\hat{g}_{i+1/2}$ is numerical flux which will be determined.
Here, we use AWENO discretization in which $\hat{g}_{i+1/2}$ are calculated by

$$\hat{g}_{i+1/2} = g_{i+1/2} + \mathcal{D}_{i+1/2}[g]. \tag{12}$$

where the first term $g_{i+1/2}$ of numerical flux in (12) is approximated by using arbitrary consistent Riemann solver $\mathcal{G}(u^-_{i+1/2}, u^+_{i+1/2})$ and second term $\mathcal{D}[g]$ of (12) is high order central difference operator:

$$\mathcal{D}[g] = -\frac{1}{24}\Delta x^2 g_{xx}|_{i+1/2} + \frac{7}{5760}\Delta x^4 g_{xxxx}|_{i+1/2}. \tag{13}$$

In sequence to achieve the fifth order accuracy in (12), the first and second terms of (13) are approximated by using central finite difference method:

$$\Delta x^2 g_{xx}|_{i+1/2} = \frac{1}{48}(-5g_{i-2} + 39g_{i-1} - 34g_i - 34g_{i+1} + 39g_{i+2} - 5g_{i+3}) + O(\Delta x^6) \tag{14}$$

$$\Delta x^4 g_{xxxx}|_{i+1/2} = \frac{1}{2}(g_{i-2} - 3g_{i-1} + 2g_i + 2g_{i+1} - 3g_{i+2} + g_{i+3}) + O(\Delta x^6) \tag{15}$$

It is well known that interpolation on conservative variables can bring false oscillations and for removing this a characteristic projection is used. The corresponding steps for this is listed below:

(i) Projection of conservative variables U_{i+k} into characteristic space through

$$\tilde{U}_{i+k} = L_{i+1/2}U_{i+k}. \qquad k = -2, ..., 3. \tag{16}$$

where the eigen-matrix $L_{i+1/2}$ is linearized by using Roe-averages in [13, 14] at each cell interface.

(ii) Now the WENO interpolation is applied on characteristic variables for computing the interface values: $\tilde{U}^-_{i+1/2}$ as:

$$\tilde{U}^-_{i+1/2} = W^-_{i+1/2}[\tilde{U}] = \sum_{k=0}^{2} w_k \tilde{U}^k(x_{i+1/2}). \tag{17}$$

that is done component-by-component.

(iii) Using right eigen-matrix the interpolated values $\tilde{U}^-_{i+1/2}$ are projected back into physical space:

$$U^-_{i+1/2} = R_{i+1/2}\tilde{U}^-_{i+1/2}. \tag{18}$$

Approximation of interface value from right is denoted by $U^+_{i+1/2}$ and can be done similarly.

Afterwords numerical flux can be calculated by

$$\hat{g}_{i+1/2} = g_{i+1/2} + \mathcal{D}[g] = \mathcal{F}(U^-_{i+1/2}, U^+_{i+1/2}) + \mathcal{D}_{i+1/2}[g]. \tag{19}$$

as formula in (12).

HIGH-ORDER WELL-BALANCED CONSERVATIVE FINITE DIFFERENCE AWENO SCHEME

Here, we discuss the high order well-balanced conservative finite difference AWENO scheme using hydrostatic reconstruction for the Ten-Moment equation. For simplicity, we take one-dimensional Ten-Moment equation for a given function W(x,y,t):

$$\frac{\partial U}{\partial t} + \frac{\partial G^x(U)}{\partial x} = S^x(U) \tag{20}$$

where vectors of conservative variables U, flux term $G^x(U)$ and the source term $S^x(U)$ are given in (2) and (3). Now we will derive one-dimensional hydrostatic steady states,i.e. assume that steady state satisfies $v_1 = v_2 = 0$. Well-balanced method is one that balances flux and source term exactly or the following condition is satisfied:

$$\partial_x G^x(u) = S^x(u) \tag{21}$$

Assuming hydrostatic solution case i.e. using $v_1 = v_2 = 0.$, we get,

$$\frac{dp_{11}}{dx} = -\frac{\rho}{2}\frac{dW}{dx}, \tag{22}$$

$$\frac{dp_{12}}{dx} = 0. \tag{23}$$

Now consider ideal gas law for Ten-Moment equation, i.e.,

$$p_{11} = \rho R T_{11} \tag{24}$$

with isothermal equilibrium (constant temperature) $T_{11} = T_{11,0}$ and ideal gas constant R.
Differentiating (24) w.r.to x we get,

$$\frac{dp_{11}}{dx} = \frac{d\rho}{dx} R T_{11,0} \tag{25}$$

Now Putting equation (25) in equation (22) and solving, we get,

$$RT_{11,0}\frac{d\rho}{dx} = -\frac{\rho}{2}\frac{dW}{dx}$$

Now integrating above relation and we get,

$$\frac{d\rho}{\rho} = -\frac{1}{2RT_{11,0}} dW$$

$$\Rightarrow ln(\rho) = -\frac{1}{2RT_{11,0}} W + C$$

$$\Rightarrow \rho = \rho_0 \, exp\left(\frac{-W}{2RT_{11,0}}\right) \qquad (where, \exp(C) = \rho_0)$$

or

$$\rho = \rho_0 a(x) \tag{26}$$

where,

$$a(x) = exp\left(\frac{-W}{2RT_{11,0}}\right)$$

Now, by ideal gas law (24), we get,

$$p_{11} = p_{11,0}\, b(x) \tag{27}$$

where,

$$p_{11,0} = \rho_0 R T_{11,0}, \qquad b(x) = a(x)$$

From (23), we get,

$$p_{12} = \zeta \tag{28}$$

where ζ is a constant.

IMPROVED SCALE-INVARIANT (SI) WENO OPERATOR

Basically, Si-property is not satisfied by the nonlinear WENO operator.

Si-operator:
For any given function $f \in L^2(\Omega)$ and scaling $\lambda \in \mathbb{R}$ on WENO operator W[.] that satisfies the Si-property

$$W[\lambda f] = \lambda W[f]$$

is called Si-operator. Generally, Si-property in [6, 13] does not satisfied by nonlinear WENO operator . Here, we use the technology of [6] to improve formula (9) as

$$w_k = \frac{\lambda_k}{\Sigma_{s=0}^{2}\lambda_s}$$
$$\lambda_k = d_k\left(1 + \left(\frac{\mu_5}{\sigma_k + \xi\theta^2}\right)^p\right) \tag{29}$$

where,

$$\theta = \varepsilon + \theta_0, \qquad \varepsilon = \frac{1}{5}\sum_{x_i \in S^5} |f_i| \tag{30}$$

and here,

$$\theta_0 = 10^{-40}, \qquad p = 2, \qquad \xi = 10^{-12}$$

For simplicity, we use $W^{\pm}_{i+1/2}[.]$ for denoting Si- WENO operator in this problem. It has been proved in [6] that

$$W^{\pm}_{i+1/2}[\lambda f] = \lambda W^{\pm}_{i+1/2}[f]. \tag{31}$$

where, λ is an arbitrary constant and f is interpolated function. And in further sections, we will observe that Si-property is fundamental for well-balanced scheme.

The main idea for well-balancing is decomposing the source term in a sum of two terms and discretise each of them separately by using finite difference methods consistent with that of flux derivatives terms approximation in conservation law.

REFORMULATION OF SOURCE TERM

The first step for well-balancing is to reformulate the source term using steady state conditions (26),(27) and (28) as discussed in [17]. So the source term in (20) can be reformulated as

$$S^x(U) = M\frac{\partial N}{\partial x} \tag{32}$$

where matrix M and vector N are as follows:

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho RT_{11,0}exp\left(\frac{W}{2RT_{11,0}}\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho v_1 RT_{11,0}exp\left(\frac{W}{2RT_{11,0}}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho v_2 RT_{11,0}exp\left(\frac{W}{2RT_{11,0}}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{33}$$

and

$$N = \begin{pmatrix} 0 \\ exp\left(\frac{-W}{2RT_{11,0}}\right) \\ 0 \\ exp\left(\frac{-W}{2RT_{11,0}}\right) \\ exp\left(\frac{-W}{2RT_{11,0}}\right) \\ 0 \end{pmatrix} \tag{34}$$

When solutions are at equilibrium states (26),(27) and (28) then M is a constant matrix. And computation of numerical derivative of N is done by AWENO scheme:

$$\frac{\partial N}{\partial x} = \frac{\hat{N}_{i+1/2} - \hat{N}_{i-1/2}}{\Delta x}, \quad i = 1, ..., N \tag{35}$$

with

$$\hat{N}_{i+1/2} = \frac{1}{2}\left(N^{-}_{i+1/2} + N^{+}_{i+1/2}\right) + \mathcal{D}_{i+1/2}(N). \tag{36}$$

Where, $N^{\pm}_{i+1/2}$ is computed by using fifth order Si-WENO interpolation as discussed above. As we have to use $W^{\pm}_{i+1/2}[.]$ for denoting Si- WENO operator in this problem. Thus,

$$W^{\pm}_{i+1/2}\left[exp\left(\frac{-W}{2RT_{11,0}}\right)\right] = exp\left(\frac{-W}{2RT_{11,0}}\right)^{\pm}_{i+1/2} \tag{37}$$

So, vector $N^{\pm}_{i+1/2}$ will be

$$N^{\pm}_{i+1/2} = \begin{pmatrix} 0 \\ exp\left(\frac{-W}{2RT_{11,0}}\right)^{\pm} \\ 0 \\ exp\left(\frac{-W}{2RT_{11,0}}\right)^{\pm} \\ exp\left(\frac{-W}{2RT_{11,0}}\right)^{\pm} \\ 0 \end{pmatrix}_{i+1/2} \tag{38}$$

In brief, semi-discretization form of conservative finite difference AWENO scheme for(20) can be written as

$$\frac{\partial U}{\partial t} + \frac{\hat{G}_{i+1/2} - \hat{G}_{i-1/2}}{\Delta x} = A\frac{\hat{N}_{i+1/2} - \hat{N}_{i-1/2}}{\Delta x} \tag{39}$$

where $\hat{G}_{i+1/2}$ is the numerical flux computed by procedure in section 4.3. Generally, at hydrostatic equilibrium state the nonlinear numerical scheme (39) is not well-balanced under the standard spatial discretization discussed above. In next subsections, by using hydrostatic reconstruction ideas discussed in [1, 2, 16] we modify the numerical flux $\hat{G}_{i+1/2}$ and the use of improved Si-WENO operator will help in achieving well-balanced property.

HYDROSTATIC RECONSTRUCTION

For getting $U^{\pm}_{i+1/2}$, we apply fifth order Si-WENO operator on the conservative variables at target point $x_{i+1/2}$. Then from $U^{\pm}_{i+1/2}$ and the ideal equation of state $\rho^{\pm}_{i+1/2}$ and $p^{\pm}_{11_{i+1/2}}$ can be computed. Now by using the ideas of hydrostatic reconstruction method, we modify the variables $\rho^{\pm}_{i+1/2}$ and $p^{\pm}_{11_{i+1/2}}$ as

$$\rho^{*,\pm}_{i+1/2} = \frac{\rho^{\pm}_{i+1/2}}{exp\left(\frac{-W}{2RT_{11,0}}\right)^{\pm}_{i+1/2}}.\pi_{max} \tag{40}$$

$$p^{*,\pm}_{11_{i+1/2}} = \frac{p^{\pm}_{11_{i+1/2}}}{exp\left(\frac{-W}{2RT_{11,0}}\right)^{\pm}_{i+1/2}}.\pi_{max}. \tag{41}$$

where,

$$\pi_{\max} = \max\left(exp\left(\frac{-W}{2RT_{11,0}}\right)^{-}_{i+1/2}, exp\left(\frac{-W}{2RT_{11,0}}\right)^{+}_{i+1/2}\right)$$

As from (28) it is observed that p_{12} is constant so

$$p^{*,\pm}_{12_{i+1/2}} = p^{\pm}_{12_{i+1/2}} = \zeta \tag{42}$$

Now, the modified vector of conservative variables,denoted by $U^{*,\pm}_{i+1/2}$ will be

$$U^{*,\pm}_{i+1/2} = \begin{pmatrix} \rho^{*,\pm} \\ \rho^{*,\pm}v_1^{\pm} \\ \rho^{*,\pm}v_2^{\pm} \\ \frac{\rho^{*,\pm}(v_1^{\pm})^2}{2} + \frac{p^{*,\pm}_{11}}{2} \\ \frac{\rho^{*,\pm}v_1^{\pm}v_2^{\pm}}{2} + \frac{p^{*,\pm}_{12}}{2} \\ \frac{\rho^{*,\pm}(v_2^{\pm})^2}{2} + \frac{p^{*,\pm}_{22}}{2} \end{pmatrix}_{i+1/2} \tag{43}$$

The numerical flux $\hat{G}_{i+1/2}$ in (19) will be modified as

$$\hat{G}^{M}_{i+1/2} = \mathcal{G}\left(U^{*,-}_{i+1/2}, U^{*,+}_{i+1/2}\right) + \begin{pmatrix} 0 \\ \frac{1}{2}(p^{-}_{11_{i+1/2}} + p^{+}_{11_{i+1/2}}) - \frac{1}{2}(p^{*,-}_{11_{i+1/2}} + p^{*,+}_{11_{i+1/2}}) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathcal{D}_{i+1/2}[G]. \tag{44}$$

Lastly, the modified semi-discretize form for conservative finite difference AWENO scheme for (20) can be written as

$$\frac{\partial U}{\partial t} + \frac{\hat{G}^{M}_{i+1/2} - \hat{G}^{M}_{i-1/2}}{\Delta x} = A\frac{\hat{N}_{i+1/2} - \hat{N}_{i-1/2}}{\Delta x}. \tag{45}$$

where $\hat{G}^M_{i+1/2}$ is numerical flux (44) and $\hat{N}_{i+1/2}$ is computed by using formula in (36).

Remark: In the frameworks of finite volume and discontinuous Galerkin methods (DG methods) in [16, 12] the hydrostatic reconstruction strategy was studied where left and right numerical fluxes $\hat{G}^l_{i+1/2}$ and $\hat{G}^r_{i+1/2}$ are modified as

$$\hat{G}^l_{i+1/2} = \mathcal{G}\left(U^{*,-}_{i+1/2}, U^{*,+}_{i+1/2}\right) + \begin{pmatrix} 0 \\ (p^-_{11_{i+1/2}} - p^{*,-}_{11_{i+1/2}}) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathcal{D}_{i+1/2}[G], \tag{46}$$

$$\hat{G}^r_{i+1/2} = \mathcal{G}\left(U^{*,-}_{i+1/2}, U^{*,+}_{i+1/2}\right) + \begin{pmatrix} 0 \\ (p^+_{11_{i+1/2}} - p^{*,+}_{11_{i+1/2}}) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathcal{D}_{i+1/2}[G]. \tag{47}$$

The modification (46) and (47) do not disturb the accuracy, these still maintains the fifth order accuracy. As, $\hat{G}_{i+1/2} - \hat{G}^\sigma_{i+1/2} = O(\Delta x^5), \quad \sigma \in \{l, r\}$ whereas in frameworks of finite difference, the modifications (46) and (47) reduces to order as

$$\frac{\hat{G}^l_{i+1/2} - \hat{G}^r_{i-1/2}}{\Delta x} = \frac{\partial G(U)}{\partial x} + O(\Delta x^4)$$

The fifth order accuracy is recovered by using (44) because the symmetries of $p_{11_{i+1/2}}$ and $p^*_{11_{i+1/2}}$ w.r.to left and right states will give a extra $O(\Delta x)$,or:

$$\frac{\hat{G}^M_{i+1/2} - \hat{G}^M_{i-1/2}}{\Delta x} = \frac{\partial G(U)}{\partial x} + O(\Delta x^5).$$

Note: The numerical $\hat{G}^M_{i+1/2}$ is used for both i-th and (i+1)-th cells. Therefore,the semi-discretized scheme (45) is fully conservative when source term vanishes.

WELL-BALANCED PROPERTY

Lets suppose that density,velocity and pressure are at hydrostatic equilibrium state (26), (27) and (28). Here, we will prove that our approach is well-balanced.

Projection of conservative variables U_{i+k} into characteristic space through the Roe-averaged left eigen-matrix $L_{i+1/2}$

$$\tilde{U}_{i+k} = L_{i+1/2} U_{i+k}$$
$$= \begin{pmatrix} 0 & -\frac{1}{2cp_{11}\rho} & 0 & \frac{1}{3p_{11}^2} & 0 & 0 \\ 0 & \frac{\sqrt{3}p_{12}}{2c\rho p_{11}} & -\frac{\sqrt{3}}{2c\rho} & -\frac{p_{12}}{p_{11}^2} & \frac{1}{p_{11}} & 0 \\ 1 & 0 & 0 & -\frac{2\rho}{3p_{11}} & 0 & 0 \\ 0 & 0 & 0 & \frac{8p_{12}^2}{3p_{11}^2} - \frac{2p_{22}}{3p_{11}} & -\frac{4p_{12}}{p_{11}} & 2 \\ 0 & -\frac{\sqrt{3}p_{12}}{2c\rho p_{11}} & \frac{\sqrt{3}}{2c\rho} & -\frac{p_{12}}{p_{11}^2} & \frac{1}{p_{11}} & 0 \\ 0 & \frac{1}{2cp_{11}\rho} & 0 & \frac{1}{3p_{11}^2} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \rho \\ 0 \\ 0 \\ \frac{p_{11}}{2} \\ \frac{p_{12}}{2} \\ \frac{p_{22}}{2} \end{pmatrix}_{i+k}$$

$$= \begin{pmatrix} \frac{1}{6p_{11}} \\ 0 \\ \frac{2\rho}{3} \\ -\frac{4p_{12}^2}{6p_{11}} + \frac{2p_{22}}{3} \\ 0 \\ \frac{1}{6p_{11}} \end{pmatrix}_{i+k} = \begin{pmatrix} \tilde{u}_1 \\ 0 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ 0 \\ \tilde{u}_4 \end{pmatrix}_{i+k} \tag{48}$$

where k=-2,...,3.

Now the WENO interpolation along with improved Si-WENO weights in (29) is applied on characteristic variables for computing the interface values: $\tilde{U}^-_{i+1/2}$ as:

$$\tilde{U}^-_{i+1/2} = W^-_{i+1/2}[\tilde{U}] = \sum_{k=0}^{2} w_k \tilde{U}^k(x_{i+1/2}). \tag{49}$$

Using right eigen-matrix$R_{i+1/2}$ the interpolated values $\tilde{U}^-_{i+1/2}$ are projected back into physical space:

$$U^-_{i+1/2} = R_{i+1/2} \tilde{U}^-_{i+1/2}$$
$$= \begin{pmatrix} \rho p_{11} & 0 & 1 & 0 & 0 & \rho p_{11} \\ -c\rho p_{11} & 0 & v_1 & 0 & 0 & c\rho p_{11} \\ -c\rho p_{12} & -\frac{c\rho}{\sqrt{3}} & v_2 & 0 & \frac{c\rho}{\sqrt{3}} & c\rho p_{12} \\ \frac{3p_{11}^2}{2} & 0 & 0 & 0 & 0 & \frac{3p_{11}^2}{2} \\ \frac{3p_{11}p_{12}}{2} & \frac{p_{11}}{2} & 0 & 0 & \frac{p_{11}}{2} & \frac{3p_{11}p_{12}}{2} \\ \frac{p_{11}p_{22}}{2} + p_{12}^2 & p_{12} & 0 & \frac{1}{2} & p_{12} & \frac{p_{11}p_{22}}{2} + p_{12}^2 \end{pmatrix} \cdot \begin{pmatrix} \tilde{u}_1^- \\ 0 \\ \tilde{u}_2^- \\ \tilde{u}_3^- \\ 0 \\ \tilde{u}_4^- \end{pmatrix}_{i+1/2}$$

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$$= \begin{pmatrix} \rho p_{11}(\tilde{u}_1^- + \tilde{u}_4^-) + \tilde{u}_2^- \\ c\rho p_{11}(-\tilde{u}_1^- + \tilde{u}_4^-) \\ c\rho p_{12}(-\tilde{u}_1^- + \tilde{u}_4^-) \\ \frac{3p_{11}^2}{2}(\tilde{u}_1^- + \tilde{u}_4^-) \\ \frac{3p_{11}p_{12}}{2}(\tilde{u}_1^- + \tilde{u}_4^-) \\ [\frac{p_{11}p_{22}}{2} + p_{12}^2](\tilde{u}_1^- + \tilde{u}_4^-) + \frac{\tilde{u}_3^-}{2} \end{pmatrix}_{i+1/2} \tag{50}$$

and by using similar procedure right cell interface $U^+_{i+1/2}$ can be computed.

Based on fact that $\tilde{u}_1 = \tilde{u}_4$ in (48). Therefore,

$$U^{\pm}_{i+1/2} = \begin{pmatrix} 2\rho p_{11}\tilde{u}_1^{\pm} + \tilde{u}_2^{\pm} \\ 0 \\ 0 \\ 3p_{11}^2\tilde{u}_1^{\pm} \\ 3p_{11}p_{12}\tilde{u}_1^{\pm} \\ p_{11}p_{22}\tilde{u}_1^{\pm} + 2p_{12}^2\tilde{u}_1^{\pm} + \frac{\tilde{u}_3^{\pm}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2\rho p_{11}W^{\pm}_{i+1/2}[\frac{1}{6p_{11}}] + W^{\pm}_{i+1/2}[\frac{2\rho}{3}] \\ 0 \\ 0 \\ 3p_{11}^2W^{\pm}_{i+1/2}[\frac{1}{6p_{11}}] \\ 3p_{11}p_{12}W^{\pm}_{i+1/2}[\frac{1}{6p_{11}}] \\ p_{11}p_{22}W^{\pm}_{i+1/2}[\frac{1}{6p_{11}}] + 2p_{12}^2W^{\pm}_{i+1/2}[\frac{1}{6p_{11}}] + \frac{1}{2}W^{\pm}_{i+1/2}[-\frac{4p_{12}^2}{6p_{11}} + \frac{2p_{22}}{3}] \end{pmatrix} \tag{51}$$

as WENO operator $W^{\pm}_{i+1/2}[.]$ used here satisfies Si-property as discossed in subsection 4.4.1.

So we can use Si-property for further deriving equation (51) as

$$U^{\pm}_{i+1/2} = \begin{pmatrix} W^{\pm}_{i+1/2}[\rho] \\ 0 \\ 0 \\ W^{\pm}_{i+1/2}[\frac{p_{11}}{2}] \\ W^{\pm}_{i+1/2}[\frac{p_{12}}{2}] \\ W^{\pm}_{i+1/2}[\frac{p_{22}}{2}] \end{pmatrix} = \begin{pmatrix} \rho^{\pm} \\ 0 \\ 0 \\ E^{\pm}_{11} \\ E^{\pm}_{12} \\ E^{\pm}_{22} \end{pmatrix}_{i+1/2} \tag{52}$$

from this we concludes

$$\rho^{\pm}_{i+1/2} = W^{\pm}_{i+1/2}[\rho] \tag{53}$$
$$p^{\pm}_{11_{i+1/2}} = W^{\pm}_{i+1/2}[p_{11}] \tag{54}$$
$$p^{\pm}_{12_{i+1/2}} = W^{\pm}_{i+1/2}[p_{12}] \tag{55}$$
$$p^{\pm}_{22_{i+1/2}} = W^{\pm}_{i+1/2}[p_{22}] \tag{56}$$

under steady state (26), (27) and (28) .

Lemma 0.1.1. *At the hydrostatic equilibrium states (26), (27) and (28), it can be shown that* $U^{*,-}_{i+1/2} = U^{*,+}_{i+1/2}$.

Proof. By solving (52) for $\rho^{\pm}_{i+1/2}$ we get

$$\begin{aligned} \rho^{\pm}_{i+1/2} &= W^{\pm}_{i+1/2}[\rho] \\ &= W^{\pm}_{i+1/2}\left[\rho_0\, exp\left(\frac{-W}{2RT_{11,0}}\right)\right] \\ &= \rho_0 W^{\pm}_{i+1/2}\left[exp\left(\frac{-W}{2RT_{11,0}}\right)\right] \end{aligned}$$

from (37), we get,

$$\rho^{\pm}_{i+1/2} = \rho_0 exp\left(\frac{-W}{2RT_{11,0}}\right)^{\pm}_{i+1/2} \tag{57}$$

and it is easy to get

$$\rho^{*,-}_{i+1/2} = \frac{\rho^{-}_{i+1/2}}{exp\left(\frac{-W}{2RT_{11,0}}\right)^{-}_{i+1/2}}.\pi_{max}$$

from (57)

$$\rho^{*,-}_{i+1/2} = \rho_0.\pi_{max} \tag{58}$$

and,

$$\rho^{*,+}_{i+1/2} = \frac{\rho^{+}_{i+1/2}}{exp\left(\frac{-W}{2RT_{11,0}}\right)^{+}_{i+1/2}}.\pi_{max}$$

$$\text{from} \quad (57)$$

$$\rho^{*,+}_{i+1/2} = \rho_0.\pi_{max} \tag{59}$$

So from (58) and (59) it can be obtained that

$$\rho^{*,-}_{i+1/2} = \rho^{*,+}_{i+1/2} \tag{60}$$

The sane procedure for $p^{\pm}_{11}$, in (54) can be applied and yields

$$\begin{aligned} p^{\pm}_{11_{i+1/2}} &= W^{\pm}_{i+1/2}[p_{11}] \\ &= W^{\pm}_{i+1/2}\left[p_{11,0}\, exp\left(\frac{-W}{2RT_{11,0}}\right)\right] \\ &= p_{11,0}W^{\pm}_{i+1/2}\left[exp\left(\frac{-W}{2RT_{11,0}}\right)\right] \end{aligned}$$

$$\text{from}(37), \text{we get,}$$

$$p^{\pm}_{11_{i+1/2}} = p_{11,0}exp\left(\frac{-W}{2RT_{11,0}}\right)^{\pm}_{i+1/2} \tag{61}$$

and it is easy to get

$$p^{*,-}_{11_{i+1/2}} = \frac{p^{-}11_{i+1/2}}{exp\left(\frac{-W}{2RT_{11,0}}\right)^{-}_{i+1/2}}.\pi_{max}$$

$$\text{from} \quad (61)$$

$$p^{*,-}_{11_{i+1/2}} = p_{11,0}.\pi_{max} \tag{62}$$

and,

$$p^{*,+}_{11_{i+1/2}} = \frac{p^{+}11_{i+1/2}}{exp\left(\frac{-W}{2RT_{11,0}}\right)^{+}_{i+1/2}}.\pi_{max}$$

$$\text{from} \quad (61)$$

$$p^{*,+}_{11_{i+1/2}} = p_{11,0}.\pi_{max} \tag{63}$$

So from (62) and (63) it can be obtained that

$$p^{*,-}_{11_{i+1/2}} = p^{*,+}_{11_{i+1/2}} \tag{64}$$

As we know that p_{12} is a constant so

$$p^{*,-}_{12_{i+1/2}} = p^{*,+}_{12_{i+1/2}} = \zeta \tag{65}$$

similarly, it can be shown that,

$$p^{*,-}_{22_{i+1/2}} = p^{*,+}_{22_{i+1/2}} \tag{66}$$

Putting the values of equation (60),(64) and (65) in equation (43), we get,

$$U^{*,-}_{i+1/2} = U^{*,+}_{i+1/2} \tag{67}$$

□

Proposition 0.1.2. *With the help of source term reformulation (32) and (33),method of hydrostatic reconstruction (43) and (44) and method of fifth order WENO interpolation with improved WENO weights (29), then modified semi-discretized scheme (45) for Ten-Moment equations is a well-balanced scheme that can maintain the steady state (26),(27) and (28) exactly.*

Proof. The modified semi-discretized scheme (45) can be reformulated as

$$\frac{\partial U}{\partial t} + \frac{(\hat{G}^M_{i+1/2} - M\hat{N}_{i+1/2}) - (\hat{G}^M_{i-1/2} - M\hat{N}_{i-1/2})}{\Delta x} = 0. \tag{68}$$

From Result 1 and consistency of Riemann solver one has following at hydrostatic equilibrium state (26),(27) and (28) that

$$\hat{G}^M_{i+1/2} = \begin{pmatrix} 0 \\ \frac{1}{2}(p^{-}_{11_{i+1/2}} + p^{+}_{11_{i+1/2}}) + \mathcal{D}_{i+1/2}[p_{11}] \\ \frac{1}{2}(p^{*,-}_{12_{i+1/2}} + p^{*,+}_{12_{i+1/2}}) + \mathcal{D}_{i+1/2}[p_{12}] \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{69}$$

and we know that p_{12} is constant,second and fourth derivatives of p_{12} will be zero, then by (13)

$$\mathcal{D}_{i+1/2}[p_{12}] = 0$$

and from (65) we have

$$p^{*,-}_{12_{i+1/2}} = p^{*,+}_{12_{i+1/2}} = \zeta$$

So, matrix $\hat{G}^M_{i+1/2}$ will be

$$\hat{G}^M_{i+1/2} = \begin{pmatrix} 0 \\ \frac{1}{2}(p^{-}_{11_{i+1/2}} + p^{+}_{11_{i+1/2}}) + \mathcal{D}_{i+1/2}[p_{11}] \\ \zeta \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{70}$$

and using (38) in (36) we get,

$$\hat{N}_{i+1/2} = \frac{1}{2}\begin{pmatrix} 0 \\ exp\left(\frac{-W}{2RT_{11,0}}\right)^{-} + exp\left(\frac{-W}{2RT_{11,0}}\right)^{+} + \mathcal{D}_{i+1/2}\left[exp\left(\frac{-W}{2RT_{11,0}}\right)\right] \\ 0 \\ exp\left(\frac{-W}{2RT_{11,0}}\right)^{-} + exp\left(\frac{-W}{2RT_{11,0}}\right)^{+} + \mathcal{D}_{i+1/2}\left[exp\left(\frac{-W}{2RT_{11,0}}\right)\right] \\ exp\left(\frac{-W}{2RT_{11,0}}\right)^{-} + exp\left(\frac{-W}{2RT_{11,0}}\right)^{+} + \mathcal{D}_{i+1/2}\left[exp\left(\frac{-W}{2RT_{11,0}}\right)\right] \\ 0 \end{pmatrix} \tag{71}$$

At hydrostatic steady state $v_1 = v_2 = 0$, matrix M will be

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho RT_{11,0} exp\left(\frac{W}{2RT_{11,0}}\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{72}$$

So with the formulas (71) and (72) we get,

$$M\hat{N}_{i+1/2} = \begin{pmatrix} 0 \\ \rho_0 RT_{11,0}\left[\frac{1}{2}\left(exp\left(\frac{-W}{2RT_{11,0}}\right)^{-} + exp\left(\frac{-W}{2RT_{11,0}}\right)^{+}\right) + \mathcal{D}_{i+1/2}\left[exp\left(\frac{-W}{2RT_{11,0}}\right)\right]\right] \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

because of Si-property of Si-WENO operator $W^{\pm}_{i+1/2}[.]$ and formula (54) we get,

$$M\hat{N}_{i+1/2} = \begin{pmatrix} 0 \\ \frac{1}{2}(p^{-}_{11_{i+1/2}} + p^{+}_{11_{i+1/2}}) + \mathcal{D}_{i+1/2}[p_{11}] \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{73}$$

So from (70) and (73), we get,

$$(\hat{G}^{M}_{i+1/2} - M\hat{N}_{i+1/2}) = \begin{pmatrix} 0 \\ 0 \\ \zeta \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{74}$$

Similarly, we get

$$(\hat{G}^{M}_{i-1/2} - M\hat{N}_{i-1/2}) = \begin{pmatrix} 0 \\ 0 \\ \zeta \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{75}$$

From (74) and (75) we get,

$$(\hat{G}^{M}_{i+1/2} - M\hat{N}_{i+1/2}) - (\hat{G}^{M}_{i-1/2} - M\hat{N}_{i-1/2}) = 0 \tag{76}$$

Using (76) in (68), we get,

$$\frac{\partial U}{\partial t} = 0 \tag{77}$$

and modified scheme (45) is well-balanced. $\square$

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