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Quasi-regularity in Right Near-rings

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ABSTRACT

Two notions viz. quasi-regular R-subgroup and radical subgroup of a right near-ring are introduced here. Again the radical of a right near-ring possesses the quasi-regular characteristic under some conditions. The concepts, under some conditions, namely quasi-regularity, radical subgroup and radical of a right near-ring are interrelated. Again a notion of quasi-radical is introduced. Again there is a relation among quasi-radical, quasi-regular as well as nilpotency in case of distributively generated right near-ring.

Keywords: Right near-ring, Distributively generated right near-ring, R-module, R-subgroup, Radical of a right near-ring, Quasi-regular R-subgroup, Radical subgroup. Nilpotent R-subgroup.

INTRODUCTION

Analogous to the concepts available in case of left near-ring discussed by Beidleman [1], Betsch [2], Laxton [7], the concepts namely quasi-regular R-subgroup, the radical and radical-subgroup of a right near-ring R with identity are introduced herein. In section 2 we show that the radical subgroup of a right near-ring R is quasi-regular R-subgroup that contains all quasi-regular left ideals. Also the radical J(R) of R is a quasi-regular ideal if and only if it is the radical-subgroup of R. In section 3 we consider distributively generated right near-rings R that satisfy d.c.c. on R-subgroups. It is shown that a quasi-regular R-subgroup of R is nilpotent. The quasi-radical of R is a quasi-regular left ideal if and only if the radical-subgroup contains all the nilpotent left ideals of R.

Definitions:

Following definitions are same as that appeared in case of left near-rings studied in [5, 6, 7, 8].

A **right near-ring** R is a system with two binary operations, addition and multiplication such that

(i) The elements of R form a group R^+ under addition.

(ii) The elements of R form a multiplicative semi-group.

(iii) $(x + y)z = xz + yz$, for all $x, y, z \in R$.

(iv) $x0 = 0$, where 0 is the additive identity of R^+ and $x \in R$.

In particular, if R contains a multiplicative semi-group S, whose elements generate R^+ and satisfy

(v) $s(y + z) = sy + sz$, for all $s \in S$ and $y, z \in R$, we say that R is a **distributively generated (d.g.) right near-ring**.

The most natural example of a right near-ring is given by the set R of identity preserving mappings of an additive group G (not necessarily abelian) into itself. If the mappings are added images and multiplication is iteration then the system (R, +, .) is a right near-ring. The right near-ring R is often called the near-ring associated with the additive group G. If S is a multiplicative semigroup of endomorphisms of G and $R^/$ is the sub-near-ring generated by S then $R^/$ is a d.g. right near-ring.

Throughout this article, R will denote a right near-ring with identity e. If we let R be a d.g. near-ring and S is a multiplicative semi-group that generates R^+ then we shall assume $e \in S$. This assumption imposes no further structural restriction on R.

An **R-module** (i.e. right near-ring module) M is a system consisting of an additive group M, a right near-ring R and a mapping $(r, m) \to rm$ of $R \times M$ into M such that for all $m \in M$ and $x, y \in R$

i) $(x + y)m = xm + ym$

ii) $(xy)m = x(ym)$

iii) em = m.
In addition, if R is a d.g. right near-ring whose additive group R^+ is generated by the multiplicative semi-group S, then we shall assume
iv) $s(m_1 + m_2) = sm_1 + sm_2$, for all $m_1, m_2 \in M$ and $s \in S$.

A simple example of a right near-ring module is an additive group over its associated near-ring.
A homomorphism f of an R-module M into an R-module $M^/$ is called an **R-homomorphism** if f(rm) = rf(m), for all $m \in M$ and all $r \in R$. Like as left near-ring appeared in [1], the **submodules** of an R-module M are defined to be kernels of R-homomorphisms. A submodule of the R-module R^+ is called a **left ideal**.

PRELIMINARIES

Betch [3] and Roth [9], in case of a left near-ring, give a necessary and sufficient condition for an additive normal subgroup of an R-module to be a submodule. We now state this result in the form of a lemma in case of right near-rings.

Lemma (1.1): An additive normal subgroup B of an R-module M is a submodule if and only if $r(m + b) - rm \in B$ for all $r \in R$, $m \in M$ and $b \in B$.

A subgroup H of an R-module M is called an **R-subgroup** if $RH = \{ rh \mid r \in R, h \in H \} \subseteq H$. The R-subgroups of the R-module R^+ are called **R-subgroups** of the right near-ring R. A submodule of an R-module M is an R-subgroup. However, of course, an R-subgroup need not be a submodule.

Let A be a non-empty subset of an R-module M. By the **R-subgroup** (resp. submodule) **generated by A** is meant the intersection of all R-subgroups (resp. submodules) that contain A. The result available in [4] in case of a left near-ring is same as that the additive subgroup of M that is generated by an arbitrary collection of submodules is a submodule. However, this is not true in general for R-subgroups.
Throughout this article we will make use of:

Lemma (1.2): Let H be an R-subgroup and B, a submodule of the R-module M. Then the additive subgroup $H + B = \{ h + b \mid h \in H, b \in B\}$ is an R-subgroup of M.

The proof of (1.2) can be found in [11] in case of left near-rings.
An R-subgroup B of a right near-ring R is called a **two-sided R-subgroup** if $BR \subseteq B$. If B is a subset of R, then (R:B) denotes the set of all elements r from R such that $rR \subseteq B$ i.e, $\{r \in R: rR \subseteq B\}$. The concept available in case of left near-ring [5] can be thought that if B is an R-subgroup of R then (R:B) is the largest two-sided R-subgroup that is contained in B. An R-subgroup B of a right near-ring R is called **nilpotent** if there exists a positive integer n such that $b_1b_2\ldots b_n = 0$ for all sequences $<b_1, b_2,\ldots, b_n{}^{,\cdots}>$ of elements from B.

Analogous to the concepts available in [1, 2, 4, 7] in case of left near-rings, the following concepts are introduced in case of right near-rings.

A left ideal B of a right near-ring R is called **modular** if B is maximal as an R-subgroup. Let I denote the collection of all modular left ideals of R. We define the **radical** of R by $J(R) = \cap B, (B \in I)$. It is understood that if I is empty, then R is its own radical, in this case R is called a **radical near-ring**. An ideal of a right near-ring is defined to be the **kernel** of a right near-ring homomorphism. A non-empty subset B of a right near-ring R is an **ideal** of R if and only if B is a left ideal and $BR \subseteq B$. The radical of a right near-ring R is an ideal.

An element q of a right near-ring R is called **quasi-regular** if and only if there is an element $r \in R$ such that $r(e - q) = e$. An R-subgroup B of R is said to be **quasi-regular** if each element of B is quasi-regular.

MAIN RESULTS

Theorem (2.1): If B is a quasi-regular R-subgroup of R then B is contained in the radical J(R) of R.
Proof: Assume that $B \not\subset J(R)$. Then there exists a modular left ideal $B^/$ such that $B \not\subset B^/$. By (1.2), $R = B^/ + B$, since $B^/$ is a maximal R-subgroup. If $e = b^/ + b$, where $b^/ \in B^/$, $b \in B$, then $e - b = b^/ \in B^/$. Now B is quasi-regular and so there is as element $r \in R$ such that $e = r(e - b) = rb^/ \in B^/$, a contradiction. Hence $B \subseteq J(R)$.

Let R be a right near-ring and B, a proper R-subgroup of R. Since R contains the identity element e, it follows by Zorn's Lemma that B is contained in a maximal R-subgroup. In particular, R contains a maximal R-subgroup. Therefore, the collection L of all maximal R-subgroups of R is non-empty.

The R-subgroup $A = \cap B$ ($B \in L$) is called the **radical-subgroup** of R.

Theorem (2.2): The radical-subgroup A of a right near-ring R is a quasi-regular R-subgroup that contains all quasi-regular left ideals R.

Proof: Let a be an element of A. We assert $R(e - a) = R$. For if $R(e - a)$ is a proper R-subgroup, then $R(e - a)$ is contained in a maximal R-subgroup B. Therefore, $e = (e - a) + a \in B$, a contradiction. Since $R(e - a) = R$ there is an element $r \in R$ such that $r(e - a) = e$, and therefore A is a quasi-regular R-subgroup.

Assume that $A^/$ is a non-zero quasi-regular left ideal of R. If $A^/$ is not contained in the radical-subgroup A, then there exists a maximal R-subgroup B such that $A^/$ is not contained in B. By (1.2), $R = A^/ + B$, since B is a maximal R-subgroup. Let $e = b + a^/$ i.e. $e - a^/ = b$ where $b \in B$, $a^/ \in A^/$. Then since $A^/$ is a quasi-regular so there exists an element $r \in R$ such that $e = r(e - a^/) = rb \in B$, a contradiction. Therefore, $A^/$ is contained in A. We have proved the following:

Corollary (2.3): The group sum of two qausi-regular left ideals of R is a quasi-regular left ideal.

Corollary (2.4): The radical J(R) of a right near-ring R is a quasi-regular ideal if and only if J(R) = A where A is the radical-subgroup of R.

Appealing to (2.2) once again we can present:
Corollary (2.5): If A is the radical-subgroup of R then (A: R) is a quasi-regular two-sided R-subgroup that contains all the quasi-regular ideals of R.

Proof: By (2.2), (A: R) is quasi-regular, since $(A: R) \subseteq A$ and A contains all quasi-regular ideals. But since (A: R) is the largest two-sided R-subgroup of A, the quasi-regular ideals of R are contained in (A: R).
From this follows:

Corollary (2.6): If $(A: R) = \{0\}$, then R contains no proper non-zero quasi-regular ideals.

3. Quasi-regularity and the radical of distributively generated right near-rings that satisfy the descending chain condition

Theorem (3.1): Let R be a d.g. right near-ring that satisfies d.c.c. on R-subgroups. If C is a quasi-regular R-subgroup, then C is nilpotent.
Proof: Let C be a non-zero quasi-regular R-subgroup. For each positive integer n, let C_n denote the R-subgroup of R that is generated by finite products of n elements from C. Hence, $C \supseteq C_1 \supseteq C_2 \supseteq \ldots$ is a decreasing sequence of R-subgroups of R and so there exists a positive integer k such that $C_k = C_{k+1} = \ldots$. Let $B = C_k$ and assume $B \neq \{0\}$. We will obtain a contradiction to this assumption. Let BoB denote the subgroup of R^+ that is generated by elements of the form b_1b_2 where $b_1, b_2 \in B$. Since R is a d.g. right near-ring, it follows from Theorem 3.3.1 of [5] in case of left near-ring that BoB is an R-subgroup. Also, $B \supseteq BoB \supseteq C_kC_k = \{ aa^/ \mid a, a^/ \in C_k \}$. Hence it follows that $B \supseteq BoB \supseteq C_{2k} = C_k = B$ and so $B = BoB$. From this we observe that there exists a minimal R-subgroup D contained in B such that $BoD \neq \{0\}$ and so there is an element $d \in D$ such that $Bd \neq \{0\}$. It is easy to verify $Bo(Bd) = (BoB)d$. Hence, $Bo(Bd) \neq \{0\}$and so $Bd = D$. Let b be an element of B such that $bd = d$. Then there is an element $r \in R$ such that $0 = r(b - e)d = d$, a contradiction. From this we are allowed to conclude $B = \{0\}$. In particular, C is nilpotent.
From (3.1) we obtain the following:

Corollary (3.2): Let R be a d.g. right near-ring that satisfies d.c.c. on R-subgroups. If the radical J(R) is a quasi-regular ideal, then it is nilpotent.

Let R is a d.g. right near-ring and $R^/$, the derived subgroup of R^+. Then since $R^/$ is a fully invariant subgroup of R^+, $R^/$ is an ideal of R. We are now ready to prove:

Theorem (3.3): Let R be a d.g. right near-ring that satisfies d.c.c. on R-subgroups. Let $R^{/}$ be quasi-regular ideal. An R-subgroup of R is nilpotent if and only if it is quasi-regular.

Proof: From (2.2), $R^{/}$ is contained in the radical-subgroup A and so every maximal R-subgroup is an additive normal subgroup of R^{+}. Hence the radical J(R) is just the radical-subgroup of R and so it is a quasi-regular ideal. Let B be a nilpotent R-subgroup. Then $B \subseteq J(R)$ by Theorem (1.5) of [7] and so it is quasi-regular.

The converse follows by (3.1).

Let R be a d.g. right near-ring that satisfies d.c.c. on R-subgroups. Let $N = \cap B$ (B, a maximal left ideal). Like as Laxton in [9] in case of a left near-ring, it is referred to N as the quasi-radical of R. It is trivial to prove N as a nilpotent left ideal that contains all nilpotent left ideals of R. We now seek conditions under which N is a quasi-regular left ideal.

Theorem (3.4): Let R be a d.g. right near-ring that satisfies d.c.c. on R-subgroups. The quasi-radical N is a quasi-regular left ideal if and only if the radical-subgroup A contains all nilpotent left ideals.

Proof: Assume that A contains all nilpotent left ideals of R. Since N is a nilpotent left ideal, $N \subseteq A$ and by (2.2) N is quasi-regular.

The converse follows from (2.2) and the fact that N contains all nilpotent left ideals.
From (3.4) we obtain:

Corollary (3.5): Let R be a d.g. right near-ring that satisfies d.c.c. on R-subgroups. The quasi-radical N is quasi-regular if and only if every nilpotent left ideal of R is quasi-regular.

Theorem (3.6): Let R be a d.g. right near-ring that satisfies d.c.c. on R-subgroups. If the quasi-radical N is contained in the radical-subgroup A, then A is left ideal if and only if N = A.

Proof: Assume that A is a left-ideal. By (2.2), (3.1) and the fact that N contains all nilpotent left ideals $A \subseteq N$.
The converse is immediate.

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