International Journal of Enhanced Research in Educational Development (IJERED)
ISSN: 2320-8708, Vol. 11 Issue 4, July-August, 2023, Impact Factor: 7.326

Fixed Point Theorems in Complete cone S-Metric Spaces

Dr. Mohini

Assistant Professor, Department of Mathematics, Adarsh Mahila Mahavidyalaya Bhiwani, Haryana, India

ABSTRACT

In this paper, Author establish some fixed point theorems in the framework of Cone S- metric spaces using implicit relation. Author extend, unify and generalize several results from current existing literature. Especially, author extend the corresponding results of Sedghi and Dung. The present work is to encouraged by its possible application, especially in discrete models for numerical analysis, where iterative schemes are extensively used due to their versatility for computer simulation. These models play an important role in applied mathematics.

Keywords: Tripled Fixed Point, S-Metric Space, Cone Metric Space.

INTRODUCTION

In 2012, Sedghi et al. introduced the concept of S-metric space which is different from other space and proved fixed point theorems in S-metric space. They also give some examples of S-metric space which shows that S-metric space is different from other spaces. In 2016, Rahman and Sarwar have discussed the fixed point results of Altman integral type mappings in S-metric spaces and in the same year Ozgur and Tas have studied new contractive conditions of integral type in complete S-metric spaces. Recently, Dhamodharan and Krishna kumar introduced the concept of Cone S-metric space and proved some fixed point theorems using various contractive conditions in the above said space .

Due to great importance of the fixed point theory, it is immensely interesting to study fixed point theorems on different concepts. Many authors studied the fixed points for mappings satisfying contractive condition in complete S-metric spaces. Popa, on the other hand, considered an implicit contraction type condition instead of the usual explicit condition. This direction of research produced a consistent literature on fixed point and common fixed point theorems in various ambient spaces. Motivated and inspired by Popa, Sedghi and Dung and others, this chapter is aimed to study and establish some fixed point theorems in setting of complete cone S-metric spaces under implicit contractive condition which is used. Following the current literature there is ample vicinity to explore and improve this new avenue of research area. Here, we prove an important result of cone S-metric space and then obtain some classical fixed point theorems as corollaries. For example, Banach's contraction mapping principle, Kannan's fixed point theorem, Chatterjae's fixed point theorems, Reich fixed point theorem and Ciric's fixed point theorem in this setting. Author results extend and generalize several results from the existing literature, especially, the results of Sedghi and Dung from complete S-metric spaces to the setting of complete cone S-metric spaces .

Definitions:

1.1 Let X be a non-empty set. A metric on X is a real function $d_\lambda : X \times X \to R$, which satisfies the following axioms:-

(i) $d_\lambda(x, y) \geq 0$ for all $x, y \in X$,

(ii) $d_\lambda(x, y) = 0$, if and only if $x = y$,

(iii) $d_\lambda(x, y) = d_\lambda(y, x)$ for all $x, y \in X$,

(iv) $d_\lambda(x, z) \leq d_\lambda(x, y) + d_\lambda(y, z)$ for all $x, y, z \in X$.

The ordered pair (X, d_λ) is called a **metric space** and $d_\lambda(x, y)$ is called the distance between x and y. The elements of X are called its points.

1.2 Let X be a non-empty set. Suppose the mapping d_λ: X x X ⟶ E (E is real Banach space) satisfies:

(i) $0 \prec d_\lambda(x, y)$ for all $x, y \in X$ and $d_\lambda(x, y) = 0$ iff $x = y$,
(ii) $d_\lambda(x, y) = d_\lambda(y, x)$ for all $x, y \in X$,
(iii) $d_\lambda(x, y) \preccurlyeq d_\lambda(x, z) + d_\lambda(z, y)$ for all $x, y, z \in X$,
then d_λ is called a **cone metric** on X and (X, d_λ) is called a **cone metric space** .

1.3 Let $\mathscr{F}: X^3 \to X$. An element (x, y, z) is called a **tripled fixed point** of $\mathscr{F}$ if $\mathscr{F}(x, y, z) = x$, $\mathscr{F}(y, x, y) = y$, $\mathscr{F}(z, y, x) = z$.

1.4 An element $(x, y, z) \in X^3$ is called a **tripled coincidence point of mapping** $\mathscr{F}: X^3 \to X$ and $\hbar : X \to X$ if $\mathscr{F}(x, y, z) = \hbar(x)$, $\mathscr{F}(y, x, y) = \hbar(y)$, $\mathscr{F}(z, y, x) = \hbar(z)$.

1.5 An element $x \in X$ is called a **common fixed point** of the mappings $\mathscr{F}: X^3 \to X$ and
: $X \to X$ if $\mathscr{F}(x, x, x) = \hbar(x) = x$.

1.6 An element $(x, y, z) \in X^3$ is called a **tripled common fixed point of mappings** $\mathscr{F}: X^3 \to X$ and $\hbar : X \to X$, if, $\mathscr{F}(x, y, z) = \hbar(x) = x$, $\mathscr{F}(y, x, y) = \hbar(y) = y$, $\mathscr{F}(z, y, x) = \hbar(z) = z$.

1.7 Suppose that E is a real Banach space, P is a cone in E with int $P \neq \emptyset$ and $\leq$ is partial ordering with respect to P. Let X be a non-empty set and let the function $S: X^3 \to E$ satisfy the following conditions:
(CSM_1) $S(x, y, z) \geq 0$;
(CSM_2) $S(x, y, z) = 0$ if and only if $x = y = z$;
(CSM_3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$, $\forall$ $x, y, z, a \in X$
Then the function S is called a cone S-metric on X and the pair (X, S) is called a cone S-metric space or simply CSMS .

Results :
2.1 Lemma

Let (X, S) be a cone S-metric space. Then, we have $S(x, x, y) = S(y, y, x)$.

2.2 Lemma

Let (X, d) be a cone metric space. Then, the following properties are satisfied:

(1) $S(u, v, z) = d(u, z) + d(v, z)$ for all $u, v, z \in X$, is a cone S-metric on X.

(2) $u_n \to u$ in (X, d) if and only if $u_n \to u$ in (X, S_d).

(3) $\{u_n\}$ is Cauchy in (X, d) if and only if $\{u_n\}$ is Cauchy in (X, S_d).

(4) (X, d) is complete if and only if (X, S_d) is complete.

2.3 Lemma

Let $f : X \to Y$ be a map from an S-metric space X to an S-metric space Y. Then, f is continuous at $x \in X$ if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.

(A_1) For all $x, y, z \in R_+$, if $y \leq \phi(x, x, y, z)$ with $z \leq 2x+y$, then $y \leq kx$.

(A_2) For all $y \in R_+$, if $y \leq \phi(y, 0, 0, y, y)$, then $y=0$.

(A_3) If $x_i \leq y_i + z_i$ for all $x_i, y_i, z_i \in R_+$, $i \leq 5$, then

$$\Phi(x_1,\ldots,x_5) \leq \Phi(y_1,\ldots,y_5) + \Phi(z_1,\ldots,z_5).$$

Moreover, for all $y \in X$, $\Phi(0,0,2y,y,0) \leq ky$.

Remarks: Note that the coefficient k in conditions (A_1) and (A_3) may be different, for example k_1 and k_3 respectively. But we assume that they are equal by taking $k=\max\{k_1, k_3\}$.

2.4 Theorem

Let T be a self-map on a complete cone S –metric space (X, S), P be a normal cone with normal constant K and

S(Tx, Tx, Ty) ≤ ϕ (S(x, x, y), S(x, x, Tx), S(y, y, Ty), S(x, x, Ty), S(y, y, Tx)) (2.1) for all x, y, ϵ X and some ϕ ϵ ψ. Then,we have

(1) If ϕ satisfies the condition (A_1), then T has a fixed point. Moreover, for any x_0 ϵ X and the fixed point x, we have

$$S(Tx_n, Tx_n, x) \leq \left(\frac{2k^n}{1-k}\right) S(x_0, x_0, Tx_0)$$

(2) If ϕ satisfies the condition (A_2) and T has a fixed point, then the fixed point is unique.

(3) If ϕ satisfies the condition (A_3) and T has a fixed point x, then T is continuous at x.

Proof : (1) for each x_0 ϵ X and n ϵ ℕ, put $x_n+1 = Tx_n$.

It follows from (2.1) and by Lemma that

$$S(x_{n+1}, x_{n+1}, x_{n+2}) = S(Tx_n, Tx_n, Tx_{n+1})$$

$$\leq \phi (S(x_n, x_n, x_{n+1}), S(x_n, x_n, Tx_n), S(x_{n+1}, x_{n+1}, Tx_{n+1}),$$

$$S(x_n, x_n, Tx_{n+1}), S(x_{n+1}, x_{n+1}, Tx_n))$$

$$= \phi (S(x_n, x_n, x_{n+1}), S(x_n, x_n, x_{n+1}), S(x_{n+1}, x_{n+1}, x_{n+2}),$$

$$S(x_n, x_n, x_{n+2}), S(x_{n+1}, x_{n+1}, x_{n+1}))$$

$$= \phi (S(x_n, x_n, x_{n+1}), S(x_n, x_n, x_{n+1}), S(x_{n+1}, x_{n+1}, x_{n+2}),$$

$$S(x_n, x_n, x_{n+2}), 0). \qquad (2.2)$$

By condition (CSM_3) and Lemma (2.1), we have

$$S(x_n, x_n, x_{n+2}) \leq 2S(x_n, x_n, x_{n+1}) + S(x_{n+2}, x_{n+2}, x_{n+1})$$

$$= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+2}) \ . \qquad (2.3)$$

Since ϕ satisfies the condition (A_1), there exists k ϵ [0, 1) such that

$$S (x_{n+1}, x_{n+1}, x_{n+2}) \leq k\ S (x_n, x_n, x_{n+1}) \leq k_{n+1}\ S(x_0, x_0, x_1) \qquad (2.4)$$

Thus, for all n < m, by using (CSM_3), Lemma (2.1) and equation (2.4), we have

$$S (x_n, x_n, x_m) \leq 2S (x_n, x_n, x_{n+1}) + S (x_m, x_m, x_{n+1})$$

$$= 2S (x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m)$$

International Journal of Enhanced Research in Educational Development (IJERED)
ISSN: 2320-8708, Vol. 11 Issue 4, July-August, 2023, Impact Factor: 7.326

$$= 2[k^n + \cdots + k^{m-1}]S(x_0, x_0, x_1)$$
$$\le \left(\frac{2k^n}{1-k}\right) S(x_0, x_0, x_1)$$

This implies that $\|S(x_n, x_n, x_m)\| \le \left(\frac{2k^n K}{1-k}\right) \|S(x_0, x_0, x_1)\|$

Taking the limit as n, m $\to \infty$, we get $\|S(x_n, x_n, x_m)\| \to 0$

Since $0 < k < 1$. Thus, we have $S(x_n, x_n, x_m) \to 0$ as n , m $\to \infty$.

This shows that the sequence $\{x_n\}$ is a Cauchy sequence in the complete cone S–metric space (X, S). By the completeness of the space, we have $\lim_{n\to\infty} x_n = x \in X$.

Moreover, taking the limit as $m \to \infty$ we get

$$S(x_n, x_n, x) \le \left(\frac{2k^{n+1}}{1-k}\right) S(x_0, x_0, x_1)$$

It implies that $S(Tx_n, Tx_n, x) \le \left(\frac{2k^n}{1-k}\right) S(x_0, x_0, Tx_0)$.

Now, we prove that x is a fixed point of T, by using the inequality (2.1) again we obtain $S(x_{n+1}, x_{n+1}, Tx) = S(Tx_n, Tx_n, Tx)$

$$\le \phi(S(x_n, x_n, x), S(x_n, x_n, Tx_n), S(x, x, Tx), S(x_n, x_n, Tx), S(x, x, Tx_n))$$

$$= \phi(S(x_n,x_n,x), S(x_n, x_n, x_{n+1}), S(x, x, Tx), S(x_n, x_n, Tx), S(x, x, x_{n+1})).$$

Note that $\phi \in \psi$,then using Lemma (2.2) and taking the limit as $n \to \infty$, we get

$$S(x, x, Tx) \le \phi(0, 0, S(x, x, Tx), S(x, x, Tx), 0) .$$

Since ϕ satisfies the condition (A_1), then $S(x, x, Tx) \le k.0 = 0$.

This shows that x= Tx. Thus, x is a fixed point of T.

(2) Let x_1, x_2 be fixed point of T. we shall prove that $x_1 = x_2$. It follows from equation (2.1) and Lemma (2.1) that

$$S(x_1, x_1, x_2) = S(Tx_1, Tx_1, Tx_2)$$

$$\le \phi(S(x_1, x_1, x_2), S(x_1, x_1, Tx_1), S(x_2, x_2, Tx_2), S(x_1, x_1, Tx_2), S(x_2, x_2, Tx_1))$$

$$= \phi(S(x_1, x_1, x_2), S(x_1, x_1, x_1), S(x_2, x_2, x_2), S(x_1, x_1, x_2), S(x_2, x_2, x_1))$$

$$= \phi(S(x_1, x_1, x_2), 0, 0, S(x_1, x_1, x_2), S(x_2, x_2, x_1))$$

$$= \phi(S(x_1, x_1, x_2), 0, 0, S(x_1, x_1, x_2), S(x_1, x_1, x_2)).$$

Since ϕ satisfies the condition (A_2), then $S(x_1, x_1, x_2) = 0$. This shows that $x_1 = x_2$. T has the fixed point of T is unique.

Let x be the fixed point of T and $y_n \to x \in X$ by Lemma (2.3),

We need to prove that $Ty_n \to Tx$. It follows from inequality (2.1) and by Lemma (2.1) that

$S(x, x, Ty_n) = S(Tx, Tx, Ty_n)$

$\leq \phi(S(x, x, y_n), S(x, x, Tx), S(y_n, y_n, Ty_n), S(x, x, Ty_n), S(y_n, y_n, Tx)$

$= \phi(S(x, x, y_n), S(x, x, x), S(y_n, y_n, Ty_n), S(x, x, Ty_n), S(y_n, y_n, x))$

$= \phi(S(x, x, y_n), 0\ S(Ty_n, Ty_n, y_n), S(x, x, y_n), S(Ty_n, Ty_n, x))$.

Since ϕ satisfies the condition (A_3), by Lemma (2.1) and (CSM3), we have

$S(Ty_n, Ty_n, y_n) \leq 2S(Ty_n, Ty_n, x) + S(y_n, y_n, x)$

$= 2S(Ty_n, Ty_n, x) + S(x, x, y_n)$

Then, we have

$S(x, x, Ty_n) \leq \phi(S(x, x, y_n), 0, 0, 0, S(x, x, y_n)) + \phi(0, 0, 2S(Ty_n, Ty_n, x),$
$S(Ty_n, Ty_n, x), 0)$

$\leq \phi((x, x, y_n), 0, 0, 0, S(x, x, y_n)) + kS(Ty_n, Ty_n, x)$

Then, By Lemma

s $= \phi(S(x, x, y_n), 0, 0, 0, S(x, x, y_n)) + kS(x, x, Ty_n)$.

Therefore,

$S(x, x, Ty_n) \leq \left(\frac{1}{1-k}\right) \phi(S(x, x, y_n), 0, 0, 0, S(x, x, y_n))$.

Note that $\phi \in \psi$, hence taking the limit as $n \to \infty$, we get $S(x, x, Ty_n) \to 0$.

This shows that $Ty_n \to x = Tx$. This completes the proof.

2.5 Corollary

Let (X, S) be a complete cone S-metric space and P be a normal cone with normal constant K. Suppose that the mapping $T : X \to X$ satisfies the following condition:

$S(Tx, Tx, Ty) \leq h\, S(x, x, y)$ for all $x, y \in X$, where $h \in [0, 1)$ is a constant. Then T has a unique fixed point in X, moreover T is continuous at the fixed point .

Proof : The assertion follows using the Theorem (5.4) with $\phi(x, y, z, s, t) = hx$ for some $h \in [0, 1)$ and all $x, y, z, s, t \in \mathbb{R}_+$.

2.6 Corollary

Let (X,S) be a complete cone S-metric space and P be a normal cone with normal constant K. Suppose that the mapping $T: X \to X$ satisfies the following condition:

$S(Tx, Tx, Ty) \leq q[S(x, x, Tx) + S(y, y, Ty)]$ for all $x, y \in X$, where $q \in [0, \frac{1}{2})$ is a constant. Then, T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof: The assertion follows using the Theorem (5.4) with $\phi(x, y, z, s, t) = q(y+z)$ for some $q \in [0, \frac{1}{2})$ and all $x, y, z, s, t \in \mathbb{R}_+$. Indeed, ϕ is continuous.

First, we have $\phi(x,x,y,z,0) = q(x+y)$. So, if $y \leq \phi(x, x, y, z, 0)$ with $z \leq 2x + y$, then y ≤ $\left(\frac{q}{1-q}\right)$ x with $\left(\frac{q}{1-q}\right) < 1$. Thus, T satisfies the condition (A_1).

Next, if y ≤ ϕ (y, 0, 0, y, y), then y = 0. Thus, T satisfies the condition (A_2).

Finally, if $x_i \le y_i + z_i$ for i ≤ 5, then

$\Phi\ (x_1, \ldots\ldots x_5) = q\ (x_2 + x_3)$

$\le q\ [(y_2 + z_2) + (y_3 + z_3)]$

$= q\ (y_2 + y_3) + q\ (z_2 + z_3)$

$= \phi\ (y_1, \ldots, y_5) + \phi\ (z_1, \ldots, z_5)$.

Moreover, ϕ (0, 0, 2y, y, 0) = q(0 + 2y) = 2qy

Where 2q < 1. Thus, T satisfies the condition (A_3).

REFERENCES

[1]. S. Sedghi, N. v. Dung, fixed point theorems on S-metric spaces, Mat. Vesnik 66, 2014.
[2]. M. A. Khamsi, Remarks on cone metric spaces and fixed point theorems of contractive mappings, Fixed point theory Appl.2010.
[3]. S. Sedghi, N. Shobe, A. Aliouche , A generalization of fixed point theorems in S-metric spaces, Mat. Vesnik 2012.
[4]. W. Shatanawi, Fixed point theory for contractive mappings saqtisfying ϕ –maps in G-metric spaces, Fixed Point theory Appl. 2010.
[5]. I. A. Bakhtin, The contraction principle in quasimetric spaces, Func. An.,Ulianowsk, Gos.Ped.Ins.30.1989.
[6]. B,C,Dhage, Generalized metric spaces mappings with fixed point Bull. calcuttaMath. Soc. 84 1992.
[7]. R. P. Agarwal, M. Meehan, D. O. Rebgan, Fixed Point Theory and Applications, Cambridge University Press, 2014.