International Journal of Enhanced Research in Science, Technology & Engineering
ISSN: 2319-7463, Vol. 7 Issue 2, February-2018, Impact Factor: 4.059

On Generalized Projection of Real von Neumann Algebras

Dr. Yassin A.S.

Department of Mathematics College of Computer Sciences and Mathematics University of Mosul

ABSTRACT

In this work, the spectral characterization of generalized projections in a prime real von Neumann algebra analogy to the work in [6] are investigated.

Keyword: real von Neumann algebra, generalized projection, orthogonal projection, normal operator

HOW TO CITE THIS ARTICLE

Dr. Yassin A.S., "On Generalized Projection of Real von Neumann Algebras", International Journal of Enhanced Research in Science, Technology & Engineering, ISSN: 2319-7463, Vol. 7 Issue 2, February-2018.

INTRODUCTION

The subject that studied and investigated here is of the theory of algebra von Neumann , spicily Jordan algebra . Let H be a complex Hilbert space and B(H) the * - algebra of all bounded linear operator on H.

Definition 1.1. : $T\in B(H)$ is called generalized projection if $T^2=T^*$, where T^* is the adjoint of T .

The notation of generalized projections on a finite dimensional Hilbert space introduced by GroB and Trenkler [5] . In this work , the concept of generalized projections is extended on a prime real von- Neumann algebra R of operators on a Hilbert space H , where H is not necessarily finite dimensional , the spectral characterization of generalized projections are obtained by using spectral theory of operators (see [9] and [7]).

Definition 1.2. : Let B(H) be * - algebra of all bounded linear operators on a Hilbert space H. A real * - algebra R in B(H) is called a real von Neumann algebra if it is closed in weak operator topology and satisfies the condition $R\cap iR=\{0\}$. The least von Neumann algebra U(R)=R+iR (complex) which contains R is called the enveloping of R. JW-algebra is a good example of a real von Neumann algebra.

We employ [1] , [2] , [8] and [10] a standard background references for the objects in this work. We recall that an algebra R is said to be prime if for ideals U and V of R with UV=0 implies either U=0 or V=0. For an operator T , the range , the null space and the spectrum of T are denoted by R(T) , N(T) and $\sigma(T)$ respectively.

By (theorems one and two [3]) we see that , if R is prime real von Neumann algebra , then U(R) is prime von Neumann algebra. Furthermore the mapping from U(R) onto R is a C-algebra isomorphism.

Definition 1.3. : Let R be prime real von Neumann algebra , an operator $T \in R$ is said to be normal if $T^*T = TT^*$, an orthogonal projection if $T^2 = T = T^*$.

2. THE SPECTRAL CHARACTERIZATION

If T is a normal operator , then there exists a unique resolution of the identity E on the Borel subset of $\sigma(T)$ such that T has the following spectral representation (see[9]) . $T = \int_{\sigma(T)} \lambda \, dE(\lambda)$

The following facts are the main results .

Theorem 2.1 : Let R be prime real von Neumann algebra and $T \in R$, then T is a generalized projection if and only if T is a normal operator and $\sigma(T) \subseteq \{0, 1, e^{i\frac{2}{3}\pi}, e^{-i\frac{2}{3}\pi}\}$. In this case , T has the following spectral representation

$$T = 0E(0) \oplus E(1) \oplus e^{i\frac{2}{3}\pi} E(e^{i\frac{2}{3}\pi}) \oplus e^{-i\frac{2}{3}\pi} E(e^{-i\frac{2}{3}\pi}) \ . \qquad \ldots(1)$$

where $E(\lambda)$ denotes the spectral projection associated with a spectral point $\lambda \in \sigma(T)$ and $E(\lambda) = 0$ if $\lambda \notin \sigma(T)$.

Proof : Let T be generalized projection , then , $T^2 = T^*$ and $TT^* = T^3 = T^2 T = T^* T$, hence T is normal operator.

Let $T = \int_{\sigma(T)} \lambda \, dE(\lambda)$, then $T^* = \int_{\sigma(T)} \overline{\lambda} \, dE(\lambda)$.

Now $T^2 = T^*$ implies that $T^2 - T^* == \int_{\sigma(T)} (\lambda^2 - \overline{\lambda}) \, dE(\lambda) = 0$.

Hence , $\lambda^2 = \overline{\lambda}$, for all $\lambda \in \sigma(T)$. If $\lambda \in \sigma(T)$ and $\lambda \neq 0$, we denote $\lambda = re^{i\theta}$, where $-\pi < \theta \leq \pi$, then $r^2 e^{2i\theta} = re^{-i\theta}$ and $r \neq 0$, so $re^{ri\theta} = e^{-2i\theta}$.

Hence , $r = 1$ and $1 = e^{-3i\theta}$. This show that $-3\theta = 2k\pi$ for an integer k , hence we obtain $-3\pi < 3\theta \leq 3\pi$, thus $k \in \{0, 1, -1\}$. If $k = -1$, then $3\theta = 2\pi$, so $\theta = \frac{2}{3}\pi$. If $k = 1$, then $3\theta = -2\pi$, so $\theta = -\frac{2}{3}\pi$. If $k = 0$, then $3\theta = 0$, so $\theta = 0$.

Therefore $\sigma(T) \subseteq \{0, 1, e^{i\frac{2}{3}\pi}, e^{-i\frac{2}{3}\pi}\}$.

Denote by $E(\lambda)$ the spectral projection of the normal operator T associated with a spectral point $\{\lambda\}$, then $E(\lambda)$, $\lambda \in \sigma(T)$ are orthogonal projections and mutual orthogonal , and

$T = 0E(0) \oplus E(1) \oplus e^{i\frac{2}{3}\pi} E(e^{i\frac{2}{3}\pi}) \oplus e^{-i\frac{2}{3}\pi} E(e^{-i\frac{2}{3}\pi})$, where $E(\lambda) \neq 0$ if $\lambda \in \sigma(T)$, $E(\lambda) = 0$ if $\lambda \in \{0, 1, e^{i\frac{2}{3}\pi}, e^{-i\frac{2}{3}\pi}\} \setminus \sigma(T)$ and $\sum_{\lambda \in \sigma(T)} \oplus E(\lambda) = I$ (see [8]).

Conversely ,assume that the operator T is normal and $\sigma(T) \subseteq \{0, 1, e^{i\frac{2}{3}\pi}, e^{-i\frac{2}{3}\pi}\}$. Then T has the following form

$T = 0E(0) \oplus E(1) \oplus e^{i\frac{2}{3}\pi} E(e^{i\frac{2}{3}\pi}) \oplus e^{-i\frac{2}{3}\pi} E(e^{-i\frac{2}{3}\pi})$, where $E(\lambda) \neq 0$ if $\lambda \in \sigma(T)$, $E(\lambda) = 0$ if $\lambda \in \{0, 1, e^{i\frac{2}{3}\pi}, e^{-i\frac{2}{3}\pi}\} \setminus \sigma(T)$ and $\sum_{\lambda \in \sigma(T)} \oplus E(\lambda) = I$. Thus

$$T^2 = 0E(0) \oplus E(1) \oplus e^{i\frac{4}{3}\pi} E(e^{i\frac{2}{3}\pi}) \oplus e^{-i\frac{4}{3}\pi} E(e^{-i\frac{2}{3}\pi})$$

$$= 0E(0) \oplus E(1) \oplus e^{-i\frac{2}{3}\pi} E(e^{i\frac{2}{3}\pi}) \oplus e^{i\frac{2}{3}\pi} E(e^{-i\frac{2}{3}\pi}) = T^* .$$

Hence T is generalized projection.

Note: Let be generalized projection , in general $\sigma(T)$ is not necessarily equal to the whole set $\{0, 1, e^{i\frac{2}{3}\pi}, e^{-i\frac{2}{3}\pi}\}$. If a number $\lambda \in \{0, 1, e^{i\frac{2}{3}\pi}, e^{-i\frac{2}{3}\pi}\}$ is not belong to $\sigma(T)$, for example $\sigma(T) = \{1, e^{i\frac{2}{3}\pi}\}$, then formula (1) has been changed by $T = E(1) \oplus e^{i\frac{2}{3}\pi} E(e^{i\frac{2}{3}\pi})$, where $E(1) \oplus E(e^{i\frac{2}{3}\pi}) = I$.

Corollary 2.2: Let R be prime real von Neumann algebra and $T \in R$ be generalized projection , then we have

(1). The range R(T) is closed .

(2). $T^4 = T$ and T^3 is an orthogonal projection on R(T) .

Proof :

(1). Since T a generalized projection , by theorem (2.1) we have that T is normal and it's spectrum is finite , so O is not a limit point of the spectrum of the normal operator T , then R(T) is closed .

(2). Clearly .

If H is a finite dimensional space , then we have the following consequence .

Corollary 2.3: Let $T \in M_{n\times n}$ be a $n \times n$ matrix . If $T^2 = T^*$, then there exists a unitary matrix $U \in M_{n\times n}$ such that UT^*U is a diagonal matrix and $UT^*U = 0I_{n_1} \oplus I_{n_2} \oplus e^{i\frac{2}{3}\pi} I_{n_3} \oplus e^{-i\frac{2}{3}\pi} I_{n_4}$, where $n = \sum_{i=1}^{4} n_i$, $0 \leq n_i \leq n$ and I_{n_i} is the identity on a suitable n_i - dimensional complex space , $i = 1, 2, 3, 4$.

Definition 2.4: Let R be prime real von Neumann algebra , by the following symbols we denote :

1. $R^{GP} = \{T \in R : T^2 = T^*\}$.

2. $R^{QP} = \{T \in R : T^4 = T\}$.

3. $R^{PI} = \{ T \in R : T$ is a partial isometry $\}$.

4. $R^{N} = \{ T \in R : TT^{*} = T^{*}T \}$.

The theorem in [4] is proved for a finite dimensional Hilbert spaces, here the same result also holds for an infinite dimensional Hilbert space the proof is different from [4] and based on the spectral representation (see [9]).

Theorem 2.5: Let R be prime real von Neumann algebra and $T \in R$, then the following statements are equivalent .

1. $T \in R^{GP}$.

2. $T \in R^{QP} \cap R^{PI} \cap R^{N}$.

3. $T \in R^{QP} \cap R^{N}$

Proof: $(1) \Rightarrow (2)$. Let $T \in R^{GP}$, then by theorem (2.1) T has the following from

$T = 0E(0) \oplus E(1) \oplus e^{i\frac{2}{3}\pi} E(e^{i\frac{2}{3}\pi}) \oplus e^{-i\frac{2}{3}\pi} E(e^{-i\frac{2}{3}\pi})$,where $E(\lambda) \neq 0$ if $\lambda \in \sigma(T)$ and $E(\lambda) = 0$ if $\lambda \in \{0, 1, e^{i\frac{2}{3}\pi}, e^{-i\frac{2}{3}\pi}\} \setminus \sigma(T)$ hence we have $T^{4} = 0E(0) \oplus E(1) \oplus e^{i\frac{8}{3}\pi} E(e^{i\frac{2}{3}\pi}) \oplus e^{-i\frac{8}{3}\pi} E(e^{-i\frac{2}{3}\pi})$

$= 0E(0) \oplus E(1) \oplus e^{i\frac{2}{3}\pi} E(e^{i\frac{2}{3}\pi}) \oplus e^{-i\frac{2}{3}\pi} E(e^{-i\frac{2}{3}\pi}) = T$.

We observe that $T^{*} = 0E(0) \oplus E(1) \oplus e^{-i\frac{2}{3}\pi} E(e^{i\frac{2}{3}\pi}) \oplus e^{i\frac{2}{3}\pi} E(e^{-i\frac{2}{3}\pi})$, then

$T^{*}T = 0E(0) \oplus E(1) \oplus E(e^{i\frac{2}{3}\pi}) \oplus E(e^{-i\frac{2}{3}\pi})$ which is an orthogonal projection on the subspace $E(1) \oplus E(e^{i\frac{2}{3}\pi}) \oplus E(e^{-i\frac{2}{3}\pi})$, hence T is a partial isometry .

Now $TT^{*} = 0E(0) \oplus E(1) \oplus E(e^{i\frac{2}{3}\pi}) \oplus E(e^{-i\frac{2}{3}\pi}) = T^{*}T$.

$\Rightarrow$ T is normal . Hence $T \in R^{QP} \cap R^{PI} \cap R^{N}$.

$(2) \Rightarrow (3)$. Clearly .

$(3) \Rightarrow (1)$. Let $T \in R^{QP} \cap R^{N}$, then T is normal and $T^{4} = T$, hence $\sigma(T) \subseteq \{\lambda : \lambda^{4} = \lambda) = \{0, 1, e^{i\frac{2}{3}\pi}, e^{-i\frac{2}{3}\pi}\}$.

Using theorem (2.1) we get $T \in R^{GP}$.

REFERENCES

[1]. Ajupov , Sh. A. , Classification and Representation of Ordered Jordan Algebras , Fan , Trshkent [Russian] , 1986.
[2]. Ajupov , Sh. A. , A new Proof of the Existence of Treces on Jordan Operator Algebras and Real von Neumann Algebras , Journal of functional Analysis , vol. 84 , No.2 Academic Press , Inc. , New York and London , 1989.
[3]. Amir A. M. , Baida S. A. , Multiplicative Primeness of Complexification for Real Algebras , J. Edu. , Vol. 18 , No. 4 , pp. 74-82 , 2006.
[4]. Baksalary J. K. , Liu X. , An Alternative Characterization of Generalized Projectors , Linear Algebra Appl. , 388 (61-65) , 2004.
[5]. GroB J. , Trenkler G. , Generalized and Hypergeneralized Projectors , Linear Algebra Appl. , 364 (463-474) , 1997.
[6]. Hong-Ke Du, Yuan Li, The Spectral Characterization of Generalized Projections, Linear Algera Appl. 400(2005) 313-318.
[7]. Kadison R.V. , Ringrose J. R. , Fundamentals of the Theory of Operator Algebras , vol. 1 , A cademic Press , 1983 .
[8]. Laurent W. M. , " An Introduction to Functional Analysis " , Waterloo , Ontario Canada N2L 3GL , 2010.
[9]. Rudin W. , Functional Analysis , 2nd ed. , McGraw-Hill , New York , 1991.
[10]. Takesaki M. , Theory of Operator Algebras I , Springer-Verlag , 1979.