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Planar Annihilating-Ideal Graph of Commutative Rings

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ABSTRACT

For a commutative ring with identity .Let AG(R) be the set of ideals of R with non-zero annihilators .The annihilating-ideal graph of R with vertex set AG(R)*= AG(R)-{0} and two distinct vertices I and J are adjacent if and only if I.J=0 .In this paper we investigate and find the graph AG(R) to be planar. Also we give some basic properties of AG(R), where R finite local rings. Finally we find planarity Zn.

Keyword: annihilating-ideal graph, planar graph, finite rings, local rings,

1-INTRODUCTION

Let R be a finite commutative ring with identity, and Z(R) (A(R)) the set of zero divisors(ideals with non-zero annihilator , respectively). We associate a simple graph Γ(R)[2](AG(R) respectively) with vertices Z(R)*=Z(R)-{0} ((A(R)*=A(R)-{0}, respectively) and two vertices x and y (I and J, respectively) are adjacent if and only if xy=0(IJ=(0), respectively).The first study of planar of zero divisor graph in 2001[3] when an interesting question was proposed by Anderson, Frazier, Lauve and Livingston: For which finite commutative rings R is Γ(R) planar? answer this question was given from by some authors see[1 ,3 , 6].Our goal in this paper is to investigate finite commutative rings whose Annihilating-ideal graph are planar. It is clear that if Γ(R) is not - planar then AG(R) is need not to be planar, for example, Γ(Z32) is shown in figure(1-1) , and figure (1-2) shows AG(Z32)

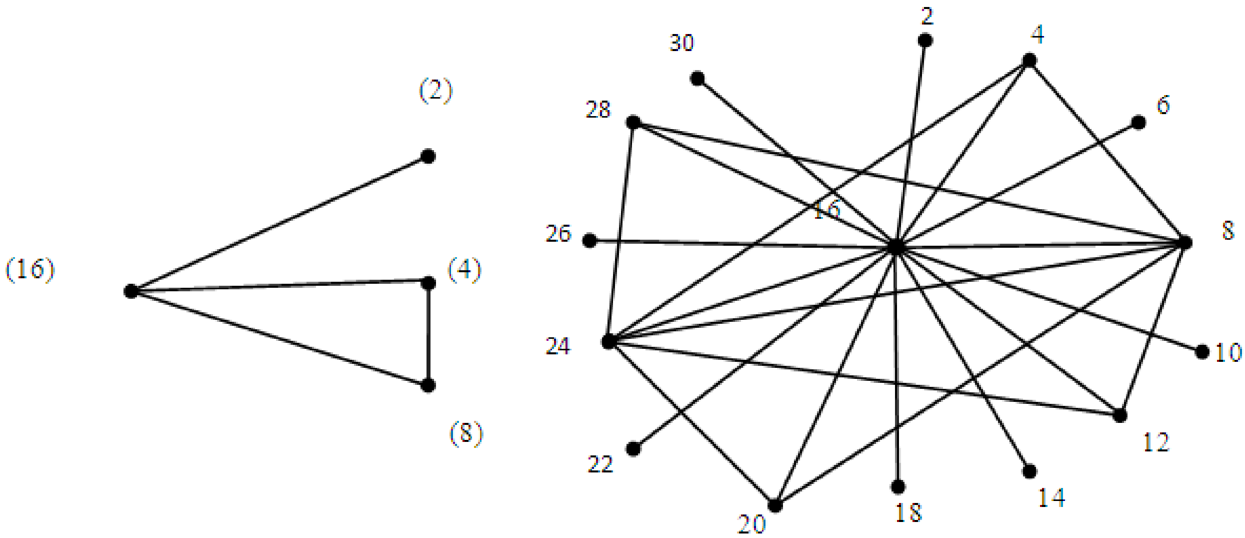

Fig. 1.1 and 1.2

For notation, we let Kn represents the complete graph on n vertices,if n=3 , then is called triangle and Km,n the complete bipartite graph with part sizes m and n. We will repeatedly use Kuratowski's theorem, which states that a graph is planar if and only if it does not contain a subdivision ofK5 orK3,3 [7].

When working with polynomial rings, say K[X]/I, we will let X denote the cosetX+I. In particular Fn is denoted by a field of order n ,$\sum_m$ is the set of coset representatives of F*m in F*=F −{0},$\sum_m^0 = \sum_m^0 \cup \{0\}$.The symble"—" is denoted the edge between two vertices ,[9]. Zn denoted the ring of integers modulo n. Finally ann(X) denoted by annihilator of se X .

2. PLANARITY OF COMMUTATIVE LOCAL RINGS.

It well known that if (R,M) is a finite local ring with maximal ideal M, then |R|=pt, for a some positive prime number p and positive integer t.In this section we investigate planarity of local rings of order pt

Question:

Under what conditionfinite commutative ring R is AG(R) is planar?
First we prove some results in a finite local rings

Proposition 2.1:

Let R be a local ring , then every minimal ideal of R adjacent with every ideal vertices in annihilating ideal graph of R .

Proof :
Let K minimal ideal of R,ifK is not adjacent with every ideal vertices then there exists an ideal vertex J of AG(R)such thatK.J $\neq 0$,so that $J \not\subseteq ann(K)$ but ann(K) maximal and R local ring which implies a contradictstherefor,K.J=0 and hence Kadjacent with every vertices in AG(R) .■

A converse of Proposition2.1 is not true in general as the following example shows:

Example 1:

Let $R \cong Z_2[X,Y]/(X^4,XY,Y^2)$ the ideals of R, $I_1=(X)$,$I_2=(Y)$,$I_3=(X,Y)$, $I_4=(X^2)$,$I_5=(X^2,Y)$, $I_6=(X+Y)$, $I_7=(X^2+Y)$, $I_8=(X^3)$, $I_9=(X^3,Y)$, $I_{10}=(X^3+Y)$ it's clearly I_9 adjacent with every other ideal vertices but not minimal ideal .

Theorem 2.2:

If R local ring with maximal ideal M and $M^2=0$, then either R has exactly one ideal M or R contains at least two minimal ideals.

Proof:

If R contains at least two minimal ideals we are done . Suppose that R contains one minimal ideal, since $M^2=0$, then $M\subseteq ann(M)$, but M maximal ideal , so that M=ann(M). On the other hand ann(M) is minimal ideal . Which leads M minimal and maximal ,since R local then for every ideal J of R,$M\subseteq J \subseteq M$ by chosen R contains one minimal ideal and hence J =M. Which implies that R contains one ideal.■

Proposition 2.3:

Let R be a local ring with $M^2=0$, then AG(R) is complete graph.

Proof:

If R contained one ideal,then AG(R) = k_1.we are done .If not ,let I and J be any ideal vertices of AG(R). Since $IJ\subseteq MM=(0)$, then any ideal vertices adjacent in AG(R).Therefore AG(R) is complete graph.■

Corollary 2.4:

Let R be a local ring with $M^2=(0)$. Then AG(R) is planar if and only if $s\leq 4$, where $s=|A(R)^*|$.

Proof:

Appling Proposition 2.3 AG(R) is complete graph so that AG(R) is planar if and only if $s\leq 4$.■

Theorem 2.5:

Let R be a local ring .Then either $M^4 = 0$ or AG(R) has a triangle

Proof:

Since R local ring , then there exists an integer $n \geq 2$ such that $M^n = 0$ and $M^{n-1} \neq 0$. Clearly $M^{n-1} . I \subseteq M^{n-1} M = 0$ for each ideal I of R. Whence M^{n-1} is adjacent to every non- zero ideal vertex I of AG(R).Now if $M^4=0$ we are done .If not , then M^2 and M^{n-2} will be adjacent, so that AG(R) hastriangular M^2—M^{n-2}—M^{n-1}—M^2.

Theorem 2.6:

Let R be a local ring with $|R|=p^t$, where p is positive prime number and t=2,3. Then R is planar

Proof:

If t=2, then by [8] $R \cong Z_{p^2}$ or $Z_p[X]/(X^2)$.So that AG(R)=K_1 which is planar. If t=3, then $R \cong F_p[X]/(X^3)$, $F[X,Y]/(X,Y)^2$, $A[X]/(PX,X^2 -ap)$ or $Z_P{}^3$,where $A \cong Z_{p^2}$ and a$\in \sum_0^2$. So that AG(R)=K_2 where $R \cong F_p[X]/(X^3)$, $A[X]/(pX,X^2)$,$Z_P{}^3$ and $AG(R) \cong K_4$,where $F[X,Y]/(X,Y)^2$, $A[X]/(PX,X^2 -ap)$,anda$\in \sum^2$. For all cases R is planar.■

Theorem 2.7:

If R is isomorphic to one of the following six rings of orderp^4
$F_p{}^2[X,Y]/(X,Y,P)^2$,$F_P[X,Y,Z]/(X,Y,Z)^2$or$F_p{}^3[X]/(X^2,PX)$,$F_p[X,Y]/(X^3,XY,Y^2)$, Z_{p^2} $[X,Y]/(X^2 - p,XY,Y^2,pX)$,$Z_{p^2}[X]/(pX,X^3)$. Then AG(R) is not planar;for all other local rings R of order p^4, AG(R) is planar.

Proof:

Consider local rings (R,M) which is not field of order p^4, where p positive prime number. In [8] Corbas and Williams conclude that the non-isomorphic commutative local ring with identity of order p^4 are precisely the following 20 rings: $F_4[X]/(X^2)$, $Z_{p^2}[X]/(X^2+X+1)$, $F_p[X]/(X^4)$, $Z_{p^2}[X]/(X^2-ap)$ where $p \neq 2$ and a$\in \sum^2$,$Z_4[X]/(X^2-2X-2)$, $Z_{p^2}[X]/(X^2-pX)$,$Z_{p^2}[X]/(X^3-p,2X)$, Z_{p^4}, $F_p[X,Y]/(X^3,XY,Y^2)$, $[X,Y]/(XY,X^2 -Y^2)$, $F_p[X,Y]/(X^2,Y^2)$, $Z_{p^2}[X,Y]/(X^2,XY- p,Y^2)$, $Z_{p^2}[X]/(X^2)$, $Z_{p^2}[X,Y]/(X^2 - p,XY,Y^2,pX)$,$Z_{p^2}[X,Y]/(X^2 - p,XY,Y^2 - p, pX)$, $Z_{p^3}[X]/(X^2-p^2,pX)$, $Z_{p^2}[X]/(pX,X3)$,$F_p{}^3[X]/(X^2,pX)$, $F_p{}^2[X,Y]/(X,Y,p)^2$ and $F_p[X,Y,Z]/(X,Y,Z)^2$.

It is easy to check that if R$\cong F_4[X]/(X^2)$ or $Z_{p^2}[X]/(X^2+X+1)$, then R has non- zero one ideal, so that AG(R) isomorphic to K_1 and hence R is planar in this cases. Consider the rings $R \cong F_p[X]/(X^4)$,$Z_{p^2}[X]/(X^2-ap)$ where $p \neq 2$ and a $\in \sum^2$,$Z_4[X]/(X^2-2X-2)$, $Z_{p^2}[X]/(X^3-p,2X)$ or Z_{p^4} , then R have non-zero three ideals and AG(R)$\cong K_{1,2}$ and hence R is planar in this cases.Consider the ring R$\cong F_p[X,Y]/(X^2,Y^2)$, $Z_{p^2}[X,Y]/(X^2,XY- p,Y^2)$,$Z_{p^2}[X]/(X^2- pX)$,or $Z_{p^2}[X]/(X^2)$, then R have non-zero five ideals and AG(R)=$K_{1,4}$, so that R is planar.

Consider the rings $R \cong F_p[X,Y]/(XY,X^2 -Y^2)$,then R have non-zero five ideals (X), (Y), (X,Y), (X^2) and (X+Y), also if$R \cong Z_{p^2}[X,Y]/(X^2 - p,XY,Y^2 - p, pX)$, then R have non-zero five ideals (2), (X), (Y), (X+Y) and (X,Y), the ring$R \cong Z_{p^3}[X]/(X^2-p^2,pX)$, then R have non-zero five ideals (2), (4), (X), (2+X) and (2,X), therefore by figure (2-1) ,AG(R) is planar. Now if $R \cong Z_p{}^3[X]/(X^2,pX)$, the ideal vertices (4), (4,X), (4+X), (X) and (2,X) are all adjacent to each other in AG(R), thus K_5 is a sub-graph og AG(R) and we get is not planar in this cases. Also if $R \cong F_p[X,Y,Z]/(X,Y,Z)^2orZ_{p^2}[X,Y]/(X,Y,P)^2$then R have non-zero eleven ideals with $M^2=0$, where M a maximal ideal in R therefore by Corollary 2.4 AG(R)=K_{11}. Whence AG(R) not planar in this cases.The ideal vertices (X^2), (X) and (X,Y) are all adjacent to (X^2+Y), (X,Y) and (Y) in AG($F_p[X,Y]/(X^3,XY,Y^2)$). The ideal vertices (2,Y), (2) and (Y) are all adjacent to (2+Y), (X) and (X,Y) in AG($Z_{p^2}[X,Y]/(X^2 - p,XY,Y^2,pX)$). Finally the ideal vertices (X^2), (p) and $(p+X^2)$ are all adjacent to (p+X), (X) and (p,X) in AG($Z_{p^2}[X]/(pX,X^3)$). Thus the last three rings all have $K_{3,3}$ as a sub-graph. Therefore are not planar.

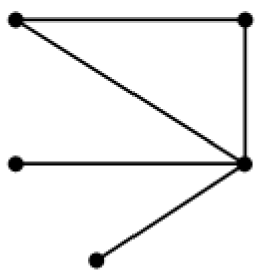

Fig (2-1)

R≅ Fp[X,Y]/(X2,Y2), Z_(p^2)[X,Y]/(X2,XY- p,Y2) ,Z_(p^2) [X]/(X^2- pX),or Z_(p^2)[X]/(X2)

Theorem 2.8:

Let $R = Z_P{}^m$ be a ring of integer module p^m where p prim and m positive number ,then R is planar iff $m \leq 8$

Proof:

Clearly $Z_P{}^m$ has (m-1) ideals, therefore (AG(R)) ≤4 if m≤ 5 implies AG(R) is planar, if m=6,7 or8 then AG(R) is planar see figures (2-2, 2-3 and 2-4) .

If m≥ 9 then the vertices ideals (p^{m-1}), (p^{m-2}), (p^{m-3}), (p^{m-4}), (p^{m-5}) adjacent so that $Z_P{}^m$ has K_5 as a sub graph ,there for $Z_P{}^m$ is not planar.

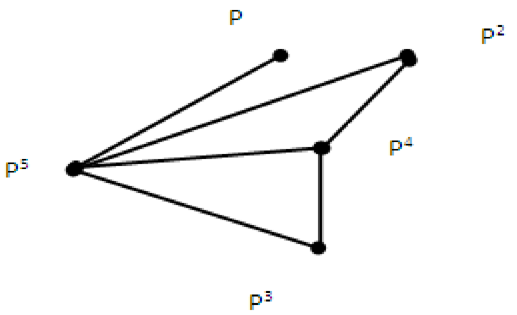

Figure2-2 AG(ZP6)

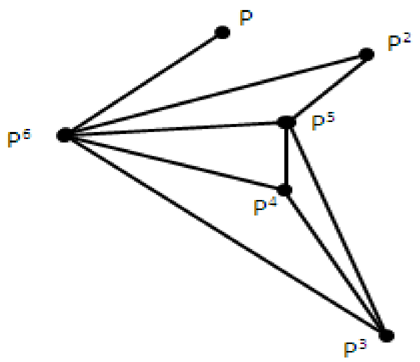

Figure 2-3 AG(ZP7)

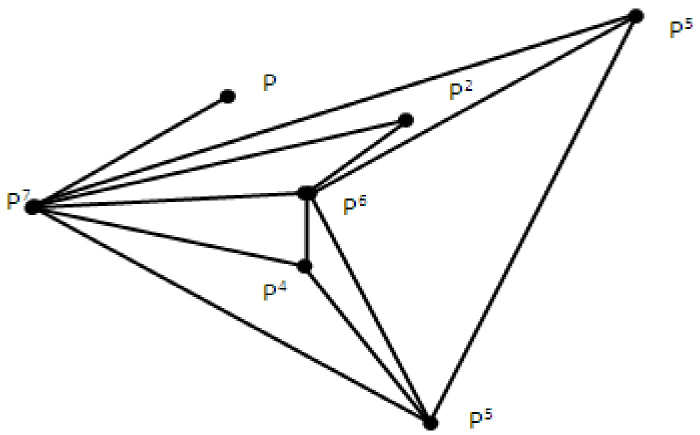

Figure 2-4 AG(ZP8)

3. PLANARITY OF COMMUTATIVE NON-LOCAL RINGS

In this section we investigate planarity of non-local rings. It well known thata finite ring R, being Artinian, is isomorphic to a finite product of Artinian local rings. Thus if R is a finite ring, then $R \cong R_1 x R_2 x \ldots x R_n$ for some n ≥ 1 and each R_i is an Artinian local ring.

Theorem 3.1

Let $R \cong R_1 x R_2 x \ldots x R_n$ for some n ≥3 and each R_i is a local ring, then R is planar if and only if $R \cong F \text{ x } F' \text{x } F''$ or $A \text{ x } F' \text{x } F''$ where F , F' and F'' are fields and A local ring contains one ideal.

Proof:

If n≥4 , then AG(R) has $K_{3,3}$ as a sub-graph by $(0,0,\ldots\ldots R_{n-1},0)$,$(0,0,\ldots.0,R_n)$,$(0,0,\ldots..,R_{n-1},R_n)$ are all adjacent to $(R_1,0,0,\ldots)$,$(R_1,R_2,0,\ldots..0)$, $(0,R_2,0,\ldots.,0)$.Then AG(R) is not planar .
If n = 3, then there exists three cases:

Case 1: if R_1 and R_2 not field , thenthere exists ideals $I_1 \subseteq R_1$and $I_2 \subseteq R_2$ such that $I_i^2 =0$, i=1,2 . Therefore AG(R) is not planar by $(R_1,0,0)$,$(I_1,0,0)$ and $(R_1,I_2,0)$ are all adjacent to $(0,I_2,0)$,$(0,I_2,R_3)$ and $(0,0,R_3)$ is $K_{3,3}$ a sub-graph of AG(R).

Case 2: If one of the R_i ,i=1,…,3 ,without loss generality say R_1 not field, then there exists two sub-cases
Sub-cases a: If R_1 has at least two ideals, say I_1 and I_2 therefore the ideal vertices $(I_1,0,0)$,$(I_2 ,0,0)$ and $(R_1,0,0)$ are all adjacent to$(0,R_2,0)$,$(0,R_2,R_3)$,$(0,0,R_3)$ } a $K_{3,3}$ sub-graph of AG(R). Therefore AG(R) not planar.
Sub-cases b:If R_1has exactly one ideal say M_1then by theorem 2.2 $M_1^2=0$.Since R_2 and R_3 fields, then R has ideals{$J_1 =(R_1,R_2,0)$, $J_2=(R_1,0,R_3)$, $J_3=(R_1,0,0)$, $J_4=(0,R_2,R_3)$, $J_5=(0,0,R_3)$,$J_6=(0,R_2,0)$,$J_7=(M_1,R_2,R_3)$,$J_8=(M_1,R_2,0)$ $J_9=(M_1,0,R_3)$,$J_{10}=(M_1,0,0)$},then AG(R) is planar see figure (3-1).

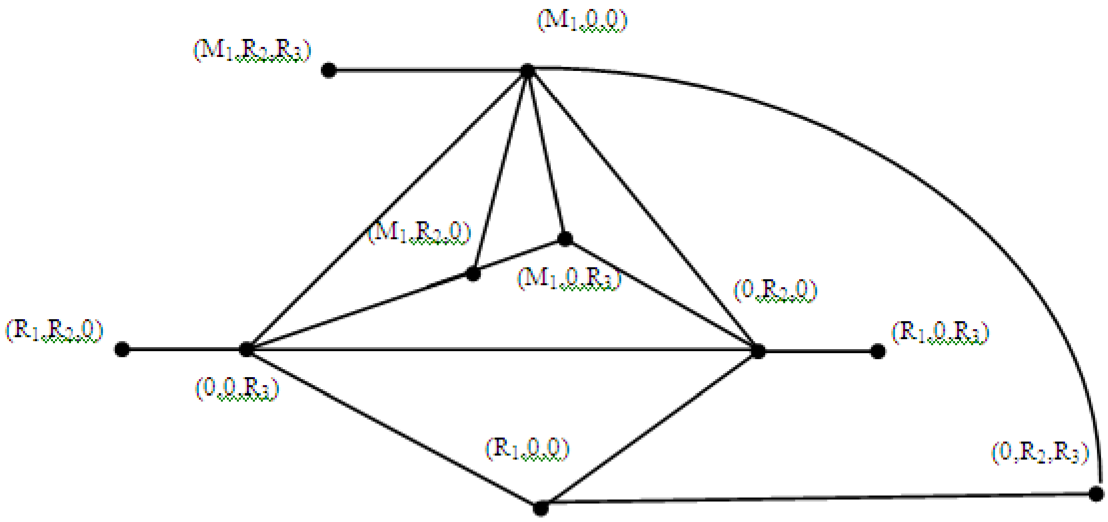

Fig (3-1)

Case 3:

IfR_1,R_2,R_3 are field, then R have ideals{$J_1=(R_1,0,0)$,$J_2=(R_1,R_2,0)$,$J_3=(R_1,0,R_3)$,$J_4=(0,R_2,0)$,$J_5=(0,R_2,R_3)$,$J_6=(0,0,R_3)$ }, whence AG(R) planar see figure (3-2).■

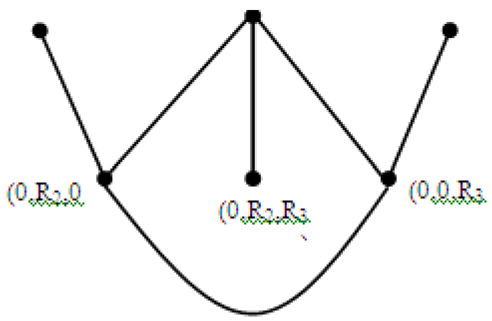

Fig. (3- 2)

Theorem 3.2:

Let R $\cong R_1 \times R_2$ where R_1 , R_2 local ring with M_1 , $M_2 \neq 0$,$M_1^2 = M_2^2 =0$, then R is planar iff R≅A x B ,where A and B local ring with one ideal.

Proof:

Since $M_1,M_2 \neq 0$,then R_1 and R_2 not field, then by theorem 2.2 R_1 and R_2 either contains one ideal or contains at least two minimal ideals .

If R_1 ,R_2 contains one ideal, then AG(R) is planar .If R_1,R_2 contains two minimal ideals say I_1,I_2 be minimal ideals in $R_1$and J_1,J_2 minimal ideals in R_2, then the ideal vertices$(R_1,0)$,$(I_1,0)$ and $(I_2,0)$are all adjacent to$(0,R_2)$,$(0,J_1)$ and$(0,J_2)$ a $K_{3,3}$ sub-graph.Therefore AG(R) not planar.

If R_1 contains one ideal and R_2 contains at least two minimal ideals, let J_1,J_2 be minimal ideals in R_2 and M_2a maximal ideal in R_2, since $J_1,J_2 \subseteq M_2$ and $J_1,J_2 \neq M_2$, we get ideals $(R_1,0)$,(R_1,J_1) and(R_1,J_2) are all adjacent to$(0,M_2)$,$(0,J_1)$,$(0,J_2)$a $K_{3,3}$ sub-graph in AG(R) and hence R is not planar.■

Theorem 3.3:

Let R be a finite ring such that $R \cong R_1 \times R_2$where R_1 and R_2 are local rings with $M_2^4 \neq 0$, then AG(R) is not planar.

Proof:

Since R_2 finite local ring, then exists an integer $n\geq 1$ such that $M_2^n=(0)$ and $M_2^{n-1}\neq 0$,but $M_2^4 \neq 0$, then we have $n \geq 5$.So that by proof of theorem 2.5 R_2 contains a triangle M_2^2— M_2^{n-1}— M_2^{n-2}— M_2^2 we note that $(M_2^{n-1})^2= M_2^{n-1}.M_2^{n-1}= M_2^n.M_2^{s1}=0$,where $s_1=n-2 > 0$,similarly $(M_2^{n-2})^2 =M_2^n .M_2^{s2}=0$, where $s_2=n-4> 0$.Therefor the ideal vertices $(R_1,0)$, (R_1,M_2^{n-1}) and (R_1,M_2^{n-2}) are all adjacent to $(0,M_2^{n-2})$, $(0, M_2^{n-1})$, $(0,M_2^2)$} a $K_{3,3}$sub-graph in AG(R) and hence R is not planar ■

Theorem 3.4:

Let $R \cong R_1 x R_2$ where R_1 and R_2 are local rings then AG(R) is planar if and only if $R \cong A_1 x A_2$ or F xB, where F is afield,A_1,A_2 are field or local rings with one ideal and B local ring contains two or three ideals with maximal ideal M,satisfies$M^2\neq 0$ and $M^4 =0$.

Proof :

It clear that, if R_1 and R_2 fields or contains one ideal, then R is planar and R_i for some i=1 or 2,contains triangular, then by Theorem 3.3R not planar. Also if R_1 and R_2 contains at least two ideals, then the ideal vertices ideals $(R_1,0)$, $(I_1,0)$ and $(I_2,0)$ are all adjacent to $(0,R_2)$,$(0,J_1)$ and $(0,J_2)$ in AG(R). Therefore $K_{3,3}$ is a sub-graph of AG(R) and therefore AG(R) not planar.So we enough investigate two cases:

Case 1: If R_1 is a field and R_2local ring contains at least four ideals say I_1, I_2, $I_3$and I_4without loss generality I_4 minimal ideal. Since R_2 local, then by Proposition 2.1 I_4 adjacent with every other ideal vertices I_1, I_2, I_3 and $I_4^2=0$. So that the ideal vertices $(R_1,0)$, (R_1,I_4) and $(0,I_4)$ are all adjacent to $(0,I_1)$, $(0,I_2)$ and $(0,I_3)$ in AG(R). Therefore $K_{3,3}$ is a sub-graph of AG(R)and so AG(R) not planar. Also if R_2 less than or equal three ideals, but not contains triangular say I_1,I_2,I_3,since R_2 not triangular $I_i^2=0$,i=1,2 and $I_1.I_2=0,I_1.I_3=0,I_2.I_3\neq 0$then the ideals in R_1xR_2 are $J_1=(R_1,I_1)$,$J_2=(R_1,I_2)$,$J_3=(R_1,I_3)$,$J_4=(R_1,0)$,$J_5=(0,R_2)$,$J_6=(0,I_1)$,$J_7=(0,I_2)$,$J_8=(0,I_3)$, then AG(R) is planarsee figure (3-3)

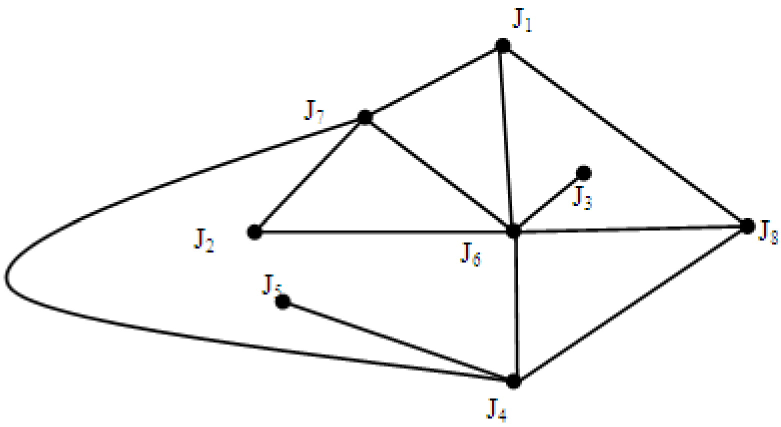

Fig. (3- 3)

Case 2: If R_1 contains one ideal say M_1, then $M_1^2=0$, Now if R_2 contains at least three ideals I_1, I_2 and I_3 with minimal ideal I_3,then the ideal vertices $(R_1,0)$, $(M_1,0)$ and (M_1,I_3) are all adjacent to $(0,I_1)(0,I_2)(0,I_3)$ in AG(R) a $K_{3,3}$. Therefore $K_{3,3}$ is a sub-graph of AG(R) and so AG(R) not planar. If R_2 contains two ideals I_1 and I_2 with $I_1^2=I_2^2=0$, then vertex ideals $(M_1,0)$, (M_1,I_1), (M_1,I_2), $(0,I_1)$ and $(0,I_2)$ is K_5 a sub-graph of AG(R), so that AG(R) not planar. If R_2 contains two ideals I_1 and I_2 with

$I_1^2=0$and $I_2^2\neq 0$, clearly $I_1.I_2=0$ then R haveideals$J_1=(R_1,0)$,$J_2=(R_1,I_1)$,$J_3=(R_1,I_2)$, $J_4=(M_1,0)$,$J_5=(M_1,I_1)$, $J_6=(M_1,I_2)$,$J_7=(M_1,R_2)$,$J_8=(0,I_1)$,$J_9=(0,I_2)$,$J_{10}=(0,R_2)$ is planar see figure (3-4)■

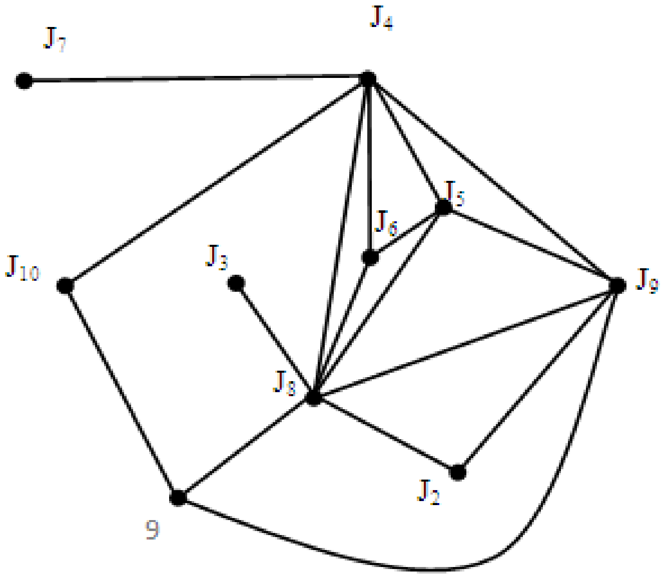

Fig. (3-4)

Theorem 3.5:

let $R \cong Z_{p1}{}^{\alpha 1}{}_{p2}{}^{\alpha 2}{}_{p3}{}^{\alpha 3} \ldots {}_{pm}{}^{\alpha m}$, where p_i distinct prime ideals and $\alpha_1, \alpha_2, \ldots, \alpha_m$ positive number $m \geq 2$ then R is planar iff $R \cong$.

Proof:

Since $Z_{p1}{}^{\alpha 1}{}_{p2}{}^{\alpha 2}{}_{p3}{}^{\alpha 3} \ldots {}_{pm}{}^{\alpha m} \cong Z_{p1}{}^{\alpha 1} x\, Z_{p2}{}^{\alpha 2} x\, Z_{p3}{}^{\alpha 3} \ldots Z_{pm}{}^{\alpha m}$
If $m \geq 3$, then by Theorem 3.1 R is planar iff $R \cong Z_{p1} x\, Z_{p2} x\, Z_{p3}$ or $Z_{p1}{}^2 x\, Z_{p2} x\, Z_{p3}$
If m= 2 , then by Theorem 3.4 R is planar iff $R \cong Z_{p1}{}^{\alpha 1} x\, Z_{p2}{}^{\alpha 2}$, where $\alpha_1, \alpha_2 = 1,2$
Or $R \cong Z_{p1} x\, Z_{p2}{}^{\alpha 2}$, where $\alpha_2 = 3,4$

Example 2:

Let $R = Z_{P1P2}{}^4$ where $p_1=3$,$p_2=2$ .Then $R \cong Z_{48}$,then R has ideals {(2),(3),(4),(6),(8),(12),(16),(24)} . Hence AG(R) is planar.

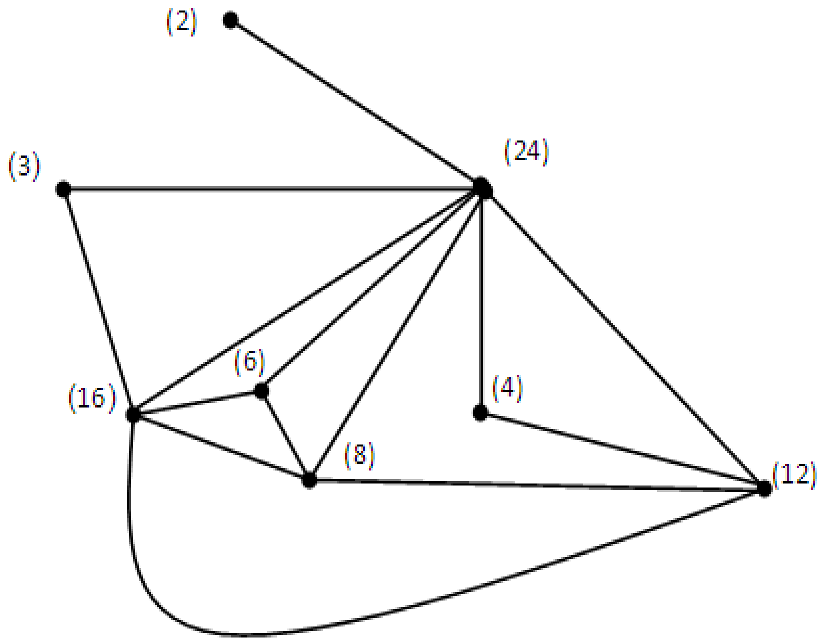

Fig. (3- 2)

REFERENCES

[1]. S .Akbari ,H .R.Maimani, and S .Yassemi " When a Zero – Divisor Graph is planar or a complete r – partite graph" . Journal of Algebra 270 (2003) 169-180
[2]. D.F.Anderson and P .S .Livingston "The Zero – Divisor Graph of a Commutative Ring" . Journal of Algebra 217 (1999) ,434-447 .
[3]. D.F.Anderson, P .S .Livingston ,A .Lauve and A . Frazier "The Zero – Divisor Graph of a Commutative Ring" II .
[4]. M .Behboodi and Z . R akeei "The Annihilating –Ideal Graph of Commutative Rings I " . (2011)
[5]. M .Behboodi and Z . R akeei "The Annihilating –Ideal Graph of Commutative Rings II " . (2011).
[6]. R . Belshoff and J .Chapman "Planar Zero – Divisor Graphs". Journal of Algebra 316 (2007) 471-480 .
[7]. J.A. Bondy, U.S.R.Murty, Graph Theory with Application ,American Elsevier, New York,1976.
[8]. B .Corbas and G.D. Williams "Ring of Order P5 .I.Nonlocal rings". Journal of Algebra 231 (2) (2000) 677-690 .