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Certain quantum calculus operators associated with the q-Analogue of Aleph-Function

Kalpana Rajput[1], Rajshreemishra[2], D.K. Jain[3], Farooq Ahmad[4], Peer Javaid Ahmad[5]

[1]Research Scholar Department of SOMAAS Jiwaji University, Gwalior, Madhya Pradesh, India. 47400
[2]Department of Mathematics, Govt. Model Science College, Gwalior (M.P.) India. 474005
[3]Department of Engineering Mathematics & Computing Madhav Institute of Technology and Science. Gwalior (M.P.) India. 474005
[4]Department of Mathematics, Govt. College for Women Nawakadal, J&K, 190001
[5]Department of Statistics, Government College for Women Nawakadal, Srinagar, Jammu & Kashmir, 190002

ABSTRACT

This paper seeks to establish the connection between the fundamental analogue of the Aleph-Function and the operators of the quantum calculus, particularly the Riemann-Louville q-integral and q-differential operators. There has also been discussion of some unique cases.Some special cases have also been discussed.

Keywords: Fractional operators& q-analogue of Aleph function.

INTRODUCTION

Kac and Cheung's book [1] entitled "Quantum Calculus" provides for the basics of so called q-calculus. More details on this type of calculus can also be found in Andrews [4,5].

Let us consider the following expression

$$\frac{\boldsymbol{f(x) - f(x_0)}}{\boldsymbol{(x - x_0)}}$$

Now letting $\boldsymbol{x \to x_0}$, we get the well - known definition of the derivative $\frac{\boldsymbol{df}}{\boldsymbol{dx}}$ of a function $\boldsymbol{f(x)}$ at $\boldsymbol{x = x_0}$. However ever , if we take $\boldsymbol{x = qx_0}$ or $\boldsymbol{x = x_0}$+ h , where q is a fixed number different from 1, and h a fixed number different from 0, and don't take the limit , we enter the fascinating world of quantum calculus . The corresponding expressions are the definitions of the q-derivative and h- derivative of $\boldsymbol{f(x)}$as defined in [1 & 2] . The same was latter on introduced by F.H.Jackson in the beginning of the twentieth century . He was the first to develop q- calculus in a systematic way.

The basic analogue of Aleph-Functiondenoted$\aleph$(z;q) for z ϵ C is defined in series form as [11]

$$\aleph^{m,n}_{p_i,q_{i};\tau_i;r}\left[\{Z,q\}\Big|^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}\right]$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j - B_j s)\prod_{j=1}^{n}\Gamma_q(1 - a_j + A_j s) z^s ds}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma_q(1 - b_{ji} + B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_q[a_{ji} - A_{ji}s]\Gamma q(s)\Gamma q(1-s)\sin\pi s]} \qquad \textbf{(1.1)}$$

The existence conditions for the integral in (1.1) are the same as for q-analogue of I-Function with r = 1.
Agrawal [3] introduced the basic analogue of the Reimann-Liouville fractional operator as follows.

$$I^{\alpha}_{q,x}f(x) = \frac{1}{\Gamma_q(\alpha)}\int_0^x (x - qt)_{\alpha-1} f(t)\, d_q(t) \; ; \text{Re}(\alpha) > 0. \qquad (1.2)$$

In particular, for f(x) = $\boldsymbol{x^p}$; we have

$$I^{\alpha}_{q,x}(x^p) = \frac{\Gamma_q(p+1)}{\Gamma_q(p+\alpha+1)} x^{p+\alpha} \; ; \; \text{Re}(\alpha) > 0. \qquad (1.3)$$

Also q-analogue of the Reimann-Liouville fractional derivative defined as [3]

$$D^{\alpha}_{q,x}f(x) = D^{n}_{q}(\, I^{n-\alpha}_{q,x} f\,)\, x \; ; \; \text{Re}(\alpha) < 0, |q| < 1$$

In particular, for f(x) = x^p ; we have

$$D^{\alpha}_{q,x}(x^p) = \frac{\Gamma_q(p+1)}{\Gamma_q(p-\alpha+1)} x^{p-\alpha} \; ; \; \text{Re}(\alpha) < 0, |q| < 1. \qquad (1.4)$$

Theorem (1.1): Let α >0 ,β >0 , γ > 0 and a ϵ R, let $\mathbf{I}_{q,x}^{\alpha}$ be the Riemann- Liouville fractional integral operator, then

$$\mathrm{I}_q^{\alpha}\left\{t^{\gamma-1}\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[\{at^{\beta},q\}\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}$$

$$= \quad x^{\gamma+\alpha-1}\aleph_{p_{i+1},q_{i+1;\tau_i;r}}^{m,n}\left\{\{ax^{\beta},q\}\Big|_{((\alpha+\gamma,\beta)(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(\gamma,\beta)(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right\}$$

Proof : To prove theorem(1.1) we apply equations (1.1) and (1.2) to the left side of theorem(1.1) we get

$$\mathbf{I}_q^{\alpha}\left\{t^{\gamma-1}\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[\{at^{\beta},q\}\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}\boldsymbol{x}$$

$$= \mathbf{I}_q^{\alpha}\left\{t^{\gamma-1}\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j-B_js)\prod_{j=1}^{n}\Gamma_q(1-a_j+A_js) \qquad x^s\mathrm{ds}}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_q[a_{ji}-A_{ji}s]\Gamma q(s)\Gamma q(1-s)sin\pi s]}\right\}$$

$$\mathbf{I}_q^{\alpha}\left\{t^{\gamma-1}\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[\{at^{\beta},q\}\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}\boldsymbol{x}$$

$$=\mathbf{I}_q^{\alpha}\left\{\sum_{k=0}^{\infty}\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j-B_js)\prod_{j=1}^{n}\Gamma_q(1-a_j+A_js)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_q[a_{ji}-A_{ji}s]\Gamma q(s)\Gamma q(1-s)sin\pi s]}\frac{\boldsymbol{a^k t^{\beta k+\gamma-1}}}{\boldsymbol{(q;q)_k}}\right\}(\boldsymbol{x})$$

$$\mathbf{I}_q^{\alpha}\left\{t^{\gamma-1}\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[\{at^{\beta},q\}\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}\boldsymbol{x}$$

$$= \left\{\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j-B_js)\prod_{j=1}^{n}\Gamma_q(1-a_j+A_js)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_q[a_{ji}-A_{ji}s]\Gamma q(s)\Gamma q(1-s)sin\pi s]}\right\}\mathbf{I}_q^{\alpha}(\boldsymbol{t^{\beta k+\gamma-1}})(\boldsymbol{x}).$$

Making the use of equation (1.3) we get

$$\mathbf{I}_q^{\alpha}\left\{t^{\gamma-1}\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[\{at^{\beta},q\}\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}\boldsymbol{x}$$

$$= \left\{\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j-B_js)\prod_{j=1}^{n}\Gamma_q(1-a_j+A_js)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_q[a_{ji}-A_{ji}s]\Gamma q(s)\Gamma q(1-s)sin\pi s]}\right\}\frac{\mathbf{a^k x^{\beta k+\gamma+\alpha-1}}}{\mathbf{(q;q)_k}}$$

$$\mathbf{I}_q^{\alpha}\left\{t^{\gamma-1}\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[\{at^{\beta},q\}\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}\boldsymbol{x}$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j-B_js)\prod_{j=1}^{n}\Gamma_q(1-a_j+A_js)\Gamma_q(\gamma+k\beta)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_q[a_{ji}-A_{ji}s]\Gamma q(s)\Gamma q(1-s)sin\pi s]\Gamma_q(\gamma+\alpha+k\beta)}\frac{\boldsymbol{a^k x^{\beta k}}}{\boldsymbol{(q;q)_k}}$$

$$\mathbf{I}_q^{\alpha}\left\{t^{\gamma-1}\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[\{at^{\beta},q\}\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}\boldsymbol{x}$$

$$= \quad x^{\gamma+\alpha-1}\aleph_{p_{i+1},q_{i+1;\tau_i;r}}^{m,n}\left\{\{ax^{\beta},q\}\Big|_{((\alpha+\gamma,\beta)(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(\gamma,\beta)(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right\}$$

This completes proof of the theorem.

Theorem (1.2): Let Re(α) < 0 , β> 0, γ > 0 and $\boldsymbol{a}$ ϵ R, let $\mathbf{D}_{q,x}^{\alpha}$ be the Riemann- Liouville fractional derivative operator, then there holds following results

$$\mathbf{D}_q^{\alpha}\left\{t^{\gamma-1}\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[\{at^{\beta},q\}\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}\boldsymbol{x} =$$

$$\boldsymbol{x}^{\boldsymbol{\gamma-\alpha-1}}\aleph_{p_{i+1},q_{i+1;\tau_i;r}}^{m,n}\left\{\{ax^{\beta},q\}\Big|_{((\alpha+\gamma,\beta)(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(\gamma,\beta)(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right\}$$

Proof : To prove theorem (1.2) we apply equations (1.1) and (1.4) to the left side of theorem (1.2) we get,

$$\mathbf{D}_q^{\alpha}\left\{t^{\gamma-1}\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[\{at^{\beta},q\}\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}\boldsymbol{x}) =$$

$$\mathbf{D}_q^{\alpha}\left\{t^{\gamma-1}\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j-B_js)\prod_{j=1}^{n}\Gamma_q(1-a_j+A_js) \qquad x^s\mathrm{ds}}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_q[a_{ji}-A_{ji}s]\Gamma q(s)\Gamma q(1-s)sin\pi s]}\right\}$$

$$\mathbf{D}_q^{\alpha}\left\{t^{\gamma-1}\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[\{at^{\beta},q\}\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}\boldsymbol{x}) =$$

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j-B_js)\prod_{j=1}^{n}\Gamma_q(1-a_j+A_js)\mathrm{ds}}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_q[a_{ji}-A_{ji}s]\Gamma q(s)\Gamma q(1-s)sin\pi s]}\mathbf{D}_q^{\alpha}(\boldsymbol{t^{\beta k+\gamma-1}})(\boldsymbol{x})$$

Making the use of equation (1.4) we get

$$\mathbf{D}_q^{\alpha}\left\{\boldsymbol{t}^{\gamma-1}\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[\{at^{\beta},q\}\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}\boldsymbol{x}) =$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma_q(b_j-B_js)\prod_{j=1}^{n}\Gamma_q(1-a_j+A_js)\mathrm{ds}}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma_q(1-b_{ji}+B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_q[a_{ji}-A_{ji}s]\Gamma q(s)\Gamma q(1-s)\sin\pi s\,]}\frac{\boldsymbol{a^k x^{\beta k+\gamma-\alpha-1}}}{\boldsymbol{(q;q)_k}}.$$

$$\mathbf{D}_q^{\alpha}\left\{\boldsymbol{t}^{\gamma-1}\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[\{at^{\beta},q\}\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}\boldsymbol{x}$$

$$= \boldsymbol{x}^{\boldsymbol{\gamma-\alpha-1}}\aleph_{p_{i+1},q_{i+1;\tau_i;r}}^{m,n}\left\{\{ax^{\beta},q\}\Big|_{((\alpha+\gamma,\beta)(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(\gamma,\beta)(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right\}$$

This completes proof of the theorem.

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