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Advancements in Techniques of Linear Programming and Its Applications

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ABSTRACT

Linear programming has proven to be an extremely powerful tool, both in modeling real-world problems and as a widely applicable mathematical theory. The study of such problems involves a diverse blend of linear algebra, multivariate calculus, numerical analysis, and computing techniques . Linear programming deals with a class of optimization problems, where both the objective function to be optimized and all the constraints, are linear in terms of the decision variables. In this paper, we discuss the developments in the field of linear programming. Also, recently developed pivot rules for linear programming are discussed in this paper. Various applications of linear programming are also discussed in this paper. Finally, we mention some suggestions for future research.

Keywords: Extremely, Optimization, Advancements, Variables, Research

INTRODUCTION

Linear programming is one of the most important areas of applied mathematical science for the last sixty years. Linear programming (LP in short), a specific case of mathematical programming, is a mathematical technique for finding a way to obtain the best possible outcome (e.g. minimum loss, maximum profit or lowest cost, etc.) in a given mathematical model for some list of requirements represented as linear relationships.

More formally, linear programming is a technique for the optimization of a linear objective function, subject to linear equality and linear constraints. Its feasible region is a convex polyhedron, which is a set defined as the intersection of finitely many half-spaces, each of which is defined by a linear inequality. Its objective function is a real-valued affine function defined on this polyhedron. A linear programming algorithm finds a point in the polyhedron where this function has the smallest (or largest) value if such a point exists.

Linear programming is a relatively new discipline in the mathematical spectrum.
Applications of the method of linear programming were first seriously attempted in the late 1930s by the Soviet mathematician Leonid Kantorovich and by the American economist Wisely Leontief in the areas of manufacturing schedules and of economics but their work was ignored for decades. However Linear programming was developed as a discipline in the 1940's, motivated initially by the need to solve complex planning problems in wartime operations. George B. Dantzig, who published the simplex method in 1947, and John von Neumann, who established the theory of duality in the same year are known as the founders of the Linear Programming. Its development accelerated rapidly after the development of simplex method in the postwar period as many industries found valuable uses for linear programming. Historical accounts of the birth and development of linear programming can be drawn from many sources, such as [6, Chapter 2] and [10]. Dantzig's personal recollections are also in [5].

A broad definition of linear programming has been given by Dantzig [5]: ―Linear programming can be viewed as part of the great revolutionary.

Development which has given mankind the ability to state general goals and to lay out a path of detailed decisions to take in order to ―best‖ achieve its goals when faced with practical situations of great complexity."

Further, Danzig [5] mentions the essential components of linear programming: ―Our tools for doing this are ways to formulate real-world problems in detailed

Mathematical terms (models), techniques for solving the models (algorithms), and engines for executing the steps of algorithms (computers and software)."

However linear programs are problems that can be expressed in canonical form:

$$\begin{aligned} &\text{maximize} && c^T x \\ &\text{subject to} && Ax \leq b \\ &\text{and} && x \geq 0 \end{aligned}$$

Where x represents the vector of variables (to be determined), c and b are known vectors coefficients, A is a (known) matrix of coefficients, and is the matrix transpose.. The T Expression to be maximized or minimized is called the objective function (cx in this case). The inequalit ies $Ax \leq b$ are the constraints which specify a convex polytope over which the objective function is to be optimized. In this context, two vectors are comparable when they have the same dimensions.

In the coming sections of the paper, we tried to give historical developments in the field of linear programming starting with the introduction of simplex method. Some applications of linear programming in the modeling real-world problems are also given in this paper.

Simplex method:
George B. Dantzig was challenged by his Pentagon colleagues to figure out how the Air Force could mechanize its planning process; to speed up the computation of deployment of forces and equipment, training and logistical support — all this during the world of desk calculators and IBM accounting equipment. George's study of Air Force requirements led him to adapt and generalize the structure behind Leontief's inter-industry model. His insight enabled him to state mathematically — for the first time — a wide class of practical and important problems that fell into the newly defined linear-programming structure. This was accomplished by July 1947 with the introduction of Simplex Method as an efficient modeling tool for practical decision making [6].

Also, in 1947, the Air Force established a major task force to work on the high-speed computation of its planning process, later named Project SCOOP (Scientific Computation of Optimal Programs), with George as chief mathematician. He stayed with Project SCOOP

The simplex method exploits the insight provided by the fundamental theorem of linear programming, which states that the optimal solution, if it exists, is at one of the vertices of the feasible polytope. Thus it reaches a solution by visiting a sequence of vertices of the polyhedron, moving from each subsequent vertex to an adjacent one characterized by a better objective function value (in the non-degenerate case). Since the number of vertices is finite, termination is guaranteed. Moreover, given the monotonic method of choosing the next vertex, the set of possible vertices decrease after each iteration, in the non-degenerate case.

Degeneracy occurs when a vertex in Rm is defined by $p > m$ constraints, and a step of length zero may be produced. In such a case, the simplex method does not actually move away from the current vertex, and thus no improvement in the objective function value can be achieved.

Efficiency of Simplex Method:
In terms of practical efficiency, the simplex algorithm has long been considered the undisputed method for solving linear programming problems. However, the simplex method has exponential complexity. It is possible that all the vertices of the feasible polyhedron have to be visited before an optimal solution is reached.

Klee and Minty [8] were the first to provide an example of pathological behavior of the simplex method. In their example, a linear program with n variables and 2n inequalities, then simplex method visits each of the 2 vertices.

However, no cases of exponential number of iterations have been encountered in real-life problems, and usually only a fraction of the vertices are actually traversed before the optimal one is found. Moreover, in most cases the simplex algorithm shows polynomial behaviour, being linear in m and sub-linear in n [7].A survey on the efficiency of the simplex method is done by Shamir [9], where a probabilistic analysis (as opposed to worst-case analysis) is also presented.

The ellipsoid method:

In 1979 a breakthrough occurred, as Khachiyan showed how to adapt the ellipsoid method for convex programming to the linear programming case, and determined the computational complexity of linear programming. In Khachiyan's ellipsoid method, the feasible polyhedron is inscribed in a sequence of ellipsoids of decreasing size. The first ellipsoid has to be large enough to include a feasible solution to the constraints; the volume of the successive ellipsoids shrinks geometrically. Therefore it generates improving iterates in the sense that the region in which the solution lies is reduced at each iteration in a monotonic fashion. The algorithm either finds a solution, as the centres of the ellipsoids converge to the optimal point, or states that no solution exists.

More details on the ellipsoid method can be found in [10] and [12], for example. The exciting property of the ellipsoid method is that it finds a solution in 2 O(nL) iterations, and thus has polynomial complexity. However, since the ellipsoid

Algorithm generally attains this worst-case bound [11], its practical performance is not competitive with other solution methods. Besides, it displays other drawbacks related to large round-off errors and a need for dense matrix computation. Nevertheless, the ellipsoid method is often used in the context of combinatorial optimization as an analytic tool to prove complexity results for algorithms [12].

The ellipsoid method guaranteed to solve any linear program in a number of steps which is a polynomial function of the amount of data defining the linear program. Consequently, the ellipsoid method is faster than the simplex method in contrived cases where the simplex method performs poorly. In practice, however, the simplex method is far superior to the ellipsoid method.

Karmarkar's Interior point method:

Interior point methods are well-suited to solving very large scale optimization problems. Their theory is well understood [20, 21] and the techniques used in their implementation are documented in extensive literature (see, for example, [23, 22]). They can be applied to a wide range of situations with no need of major changes. In particular, they have been successfully applied to complementarity problems, quadratic programming, convex nonlinear programming, second-order cone programming and semi-definite programming.

The main idea behind interior point methods is fundamentally different to the one that inspires the simplex algorithm. Interior point methods do not pass from vertex to vertex, but pass only through the interior of the feasible region. Though this property is easy to state, the behavior of interior-point methods is much less easily understood than that of the simplex method. Therefore, by embedding the linear problem in a nonlinear context, an interior point method escapes the ―curse of dimensionality" characteristic of dealing with the combinatorial features of the linear programming problem. Interior-point methods are now generally considered competitive with the simplex method.

In 1984, Narendra Karmarkar introduced an interior-point method for linear programming, combining the desirable theoretical properties of the ellipsoid method and practical advantages of the simplex method [14]. After the introduction of this algorithm, interior point methods have attracted the interest of a growing number of researchers. This algorithm was also proved to have polynomial complexity: indeed, it converges in O(nL) iterations. As opposed to Khachiyan's ellipsoid method, in practice Karmarkar's algorithm actually performs much better than its worst-case bound states.For details on Karmarkar's algorithm, we refer to [24].

Karmarkar [14] explained the advantage of an interior point approach as follows:―In the simplex method, the current solution is modified by introducing a nonzero coeffcient for one of the columns in the constraint matrix. Our method allows the current solution to be modifed by introducing several columns at once."

Karmarkar announced that his method was extremely successful in practice, claiming to beat the simplex method by a large margin (50 times, as reported in [13]). A variant of Karmarkar's original algorithm was then proposed and implemented by Adler, Resende, Veiga and Karmarkar [15]. Since then, the theoretical understanding has considerably improved, many algorithmic variants have been proposed and several of them have shown to be computationally viable alternatives to the simplex method.

Year	**Name of Mathematician**	**Contribution in the field of LinearProgramming**
1762	**Lagrange**	**Solved tractable optimization problems With simple equality constraints.**

1820	**Gauss**	**Solved linear system of equations (Gauss elimination Method)**
1866	**Wilhelm Jordan**	**Refined the method to finding least squared errors as a measure of goodness-of-fit (Gauss-Jordan Method)**
1947	**Dantzig**	**Simplex Method**
1968	**Fiacco and McCormick**	**Interior Point Method**
1969	**S. Zionts**	**Criss-Cross method**
1969	**R. H. Bartels and G. H.Golub**	**The simplex method of linear programming using the LU decompostion**
1977	**RG Bland**	**New finite pivoting rules for thesimplex method**
1979	**Khachiyan**	**ellipsoid method**
1982	**J. K Reid**	**A sparsity-exploiting variant of the Bartels-Golub decomposition Method**
1984	**Karmarkar**	**A New Polynomial Time Algorithm for Linear Programming**
1997	**Z Wang**	**A conformal elimination-free algorithm**
2003	**J. Gondzio & A. Grothey**	**Reoptimization with the primal-dual interior point method**
2003	**J. Gondzio & R. Sarkissian**	**Parallel interior-point solver for structured linear programs.**
2003	**Paparrizos, Samaras & Stephanides**	**A new efficient primal dual simplex algorithm**
2005	**A. Oliveira & C. Sorensen**	**New class of preconditioners for large-scale linear systems from interiorpoint methods for LP**

There are classes of problems that are best solved with the simplex method, and others for which an interior point method is preferred. Size, structure and sparsity play a major role in the choice of algorithm for computations. As a rule of thumb, with the increase of problem dimension, interior point methods become more effective. However, this does not hold in the hyper-sparse case, where the simplex method is virtually unbeatable [18, 19], and for network problems, where the specialised network simplex method can exploit the structure in an extremely efficient manner [12].

Pivot Rules:
The geometrical operation of moving from a basic feasible solution to an adjacent basic feasible solution is implemented as a pivot operation. Selection of pivots in LP algorithms plays a very significant role in the performance of an algorithm.Following work by Zionts[26] and Bland[27] researchers have developed new methods for selecting pivots in LP solution algorithms. Ultimately this has led to methods which solve LP problems without requiring feasibility of the basis. The so-called criss-cross method has attracted some attention. A finite crises-cross algorithm, combining aspects of the work of Zionts and Bland has been developed independently by Chang[28], Terlaky[29] and Wang[30]. Because feasibility of the basis is not required, a crises-cross method can be regarded as different from Simplex type methods. A survey on pivot algorithms in general can be found in Terlaky and Zhang[31]. The crises-cross method selects a pivot element from a row and column without resorting to any type of ratio test. Instead criteria such as smallest-subscript, first-in-last-out/last-out-first- in, or most-often-selected-variable are used. The ideas used in crises-cross methods have been inspired by work on matroids and show promise.

Applications in various domains:
It would be impossible in this review to provide comprehensive detail on all the many applications of LP that have been published over the years. Linear programming is one of the most widely applied methodologies. Around

85% of Fortune 500 companies had used LP in various domains of industries. In industrial environments, various techniques of LP are used in Corporate Planning, Factory Planning, Product Distribution,

Lease-Buy Decision, Production Scheduling, Inventory Control and Workman's Compensation. In the field of agriculture, LP techniques can be used for Food Manufacturing, Deport Location and Irrigation System. Other applications of linear programming methods include manpower planning, Activity Planning, Accounting & Finance, Administration, Education, & Politics, Advertising and Marketing, Allocation of Financial Budgets and Capital Investment.

FUTURE SCOPE

Efficient Linear programming algorithms can be developed by combining both primal and dual paths for reducing the duality gap and converging to the optimal solution much faster. Linear-fractional programming (LFP),a generalization of LP, have a lot of possibilities of improvement. Interior point methods remains an active and fruitful area of research. Integration of Interior- point methods to develop hybrid algorithms for solving very large size LP for real- world applications by exploring: special structure, sparsely, decomposition, parallel computation will remain an area of future research. Non-linear optimization with linear constraints, Conic Programming, Semi-definite Programming is still having lot of possibilities.

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