International Journal of Enhanced Research in Science, Technology & Engineering
ISSN: 2319-7463, Vol. 12 Issue 3, March-2023, Impact Factor: 7.957

Fusion Classes of non-abelian metabelian groups of order upto 24

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ABSTRACT

In this paper, G be the non-abelian metabelian group and $\overline{cl(a)}$ denotes the fusion class of element a in G. Fusion class is an equivalence relation. A group is metabelian group if its commutator group is abelian or if it has an abelian normal subgroupin which the factor group is also abelian.There are 35 non-Abelian metabelian groups of order less than or equal to 24.In this paper, we investigate the fusion classes of all non-abelian metabelian groups of order less than or equal to 24andthe verification has been made through GAP(Groups Algorithm Programming) software.

Keywords: Metabelian Groups, Automorphism, Fusion Classes.

INTRODUCTION

Let G be a finite metabelian group, Z_n denotes the cyclic group of order n, S_n denotes the permutation group of degree n, D_n denotes the dihedral group of order $2n$, Q_n denotes quatornion group. A group G is said to be metabelian if G', the derived subgroup of G is abelian. Much has been investigated about the properties of metabelian groups in the literature. In paper [3], structure of metabelian groups of order upto 24 has been described. In paper [4], the authors studied about the conjugacy classes of metabelian groups of order less than 24. In paper [1], [2], automorphisms of some non-abelian groups of order $p^4$are computed. In the present paper, we shall find the fusion classes of metabelian non-abelian groups of order less than or equal to 24.

In [3], Rehman Abdul described the metabelian groups of order less than or equal to 24.

$$(1)D_3 \cong S_3 \cong < a,b; a^3 = b^2 = 1, b\,a\,b = a^{-1} >.$$
$$(2)D_4 \cong < a,b; a^4 = b^2 = 1, b\,a\,b = a^{-1} >.$$
$$(3)Q_3 \cong < a,b: a^4 = 1, b^2 = a^2, a\,b\,a = b >.$$
$$(4)D_5 \cong < a,b: a^5 = b^2 = 1, b\,a\,b = a^{-1} >.$$
$$(5)Z_3 \rtimes Z_4 \cong < a,b: a^3 = b^4 = 1, b^{-1}a\,b = a^2 >.$$
$$(6)A_4 \cong < a,b,c: a^2 = b^2 = c^3 = 1, b\,a = a\,b, c\,a = a\,b\,c, c\,b = a\,c >.$$
$$(7)D_6 \cong < a,b: a^6 = b^2 = 1, b\,a\,b = a^{-1} >.$$
$$(8)D_7 \cong < a,b: a^7 = b^2 = 1, b\,a\,b = a^{-1} >.$$
$$(9)D_8 \cong < a,b: a^8 = b^2 = 1, b\,a\,b = a^{-1} >.$$
$$(10)G \cong < a,b: a^8 = b^2 = 1, b\,a\,b = a^3 >.$$
$$(11)Q_{16} \cong < a,b: a^8 = 1, \quad a^4 = b^2, \quad a\,b\,a = b >.$$
$$(12)D_4 \times Z_2 \cong < a,b,c: a^4 = b^2 = c^2 = 1, a\,c = c\,a, b\,c = c\,b, b\,a\,b = a^{-1} >.$$
$$(13)Q_3 \times Z_2 \cong < a,b,c: a^4 = b^4 = c^2 = 1, b^2 = a^2, b\,a = a^3b, a\,c = c\,a, b\,c = c\,b >.$$
$$(14)Modular - 16 = G \cong < a,b: a^8 = b^2 = 1, a\,b = b\,a^5 >.$$
$$(15)B \cong < a,b: a^4 = b^4 = 1, a\,b = b\,a^3 >.$$
$$(16)K \cong < a,b,c: a^4 = b^2 = c^2 = 1, a\,b = b\,a, a\,c = c\,a, c\,b = a^2b\,c >.$$
$$(17)G_{4,4} \cong < a,b: a^4 = b^4 = (a\,b\,)^2 = 1, a\,b^3 = b\,a^3 >.$$
$$(18)D_9 \cong < a,b: a^9 = b^2 = 1, b\,a\,b = a^{-1} >.$$
$$(19)S_3 \times Z_3 \cong < a,b,c: a^3 = b^2 = c^3 = 1, b\,a = a^{-1}b, a\,c = c\,a, b\,c = c\,b >.$$
$$(20)(Z_3 \times Z_3) \rtimes Z_2 \cong < a,b,c: a^2 = b^3 = c^3 = 1, b\,c = c\,b, b\,a\,b = a, c\,a\,c = a >.$$
$$(21)D_{10} \cong < a,b: a^{10} = b^2 = 1, b\,a\,b = a^{-1} >.$$
$$(22)Fr_{20} \cong Z_5 \rtimes Z_4 \cong < a,b: a^4 = b^5 = 1, b\,a = a\,b^2 >.$$
$$(23)Z_5 \rtimes Z_4 \cong < a,b: a^4 = b^5 = 1, b\,a\,b = a >.$$

International Journal of Enhanced Research in Science, Technology & Engineering
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$$(24) Fr_{21} \cong Z_7 \rtimes Z_3 \cong < a, b: a^3 = b^7 = 1, b\, a = a\, b^2 >.$$
$$(25) D_{11} \cong < a, b: a^{11} = b^2 = 1, b\, a\, b = a^{-1} >.$$
$$(26) S_3 \times Z_4 \cong < a, b, c: a^3 = b^2 = c^4 = 1, a^b = a^{-1}, a\, c = c\, a, b\, c = c\, b >.$$
$$(27) S_3 \times Z_2 \times Z_2 \cong < a, b, c, d: a^3 = b^2 = c^2 = d^2 = 1, a^b = a^{-1}, a\, c = c\, a, a\, d = d\, a, b\, c = c\, b, b\, d = d\, b, c\, d = d\, c >.$$
$$(28) D_4 \times Z_3 \cong < a, b, c: a^3 = b^4 = c^2 = 1, b^c = b^{-1}, a\, b = b\, a, a\, c = c\, a >.$$
$$(29) Q \times Z_3 \cong < a, b, c: a^4 = c^3 = 1, a^2 = b^2, a^b = a^{-1}, b\, c = c\, b, a\, c = c\, a >.$$
$$(30) A_4 \times Z_2 \cong < a, b, c, d: a^2 = b^2 = c^3 = d^2 = 1, a\, b = b\, a, c\, a = a\, b\, c, a\, d = d\, a, a\, c = c\, b, b\, d = d\, b, c\, d = d\, c >.$$
$$(31) Q_{12} \cong < a, b: a^{12} = 1, a^6 = b^2, b^{-1} a\, b = a^{-1} >.$$
$$(32) D_{12} \cong < a, b: a^{12} = b^2 = 1, b\, a\, b = a^{-1} >.$$
$$(33) Z_6 \rtimes Z_4 \cong < a, b: a^4 = b^6 = 1, b\, a\, b = a >.$$
$$(34) Z_3 \rtimes Z_8 \cong < a, b: a^3 = b^8 = 1, b\, a\, b^{-1} = a^{-1} >.$$
$$(35) Z_3 \rtimes Q \cong < a, b, c: a^2 = b^6 = c^2 = 1, ab = ba, \quad ac = ca, (cb)^2 = 1 >.$$

Fusion classes of all non-abelian metabelian groups of order less than equal to 24
Fusion classes of groups of order 6

Fusion Classes of D_3:

$$D_3 \cong S_3 \cong < a, b; a^3 = b^2 = 1, bab = a^{-1} >$$

Fusion Classes are:

$$\overline{cl(1)} = \{1\}$$
$$\overline{cl(a)} = \{a, a^2\}$$
$$\overline{cl(b)} = \{ab, a^2b, b\}$$

So there are 3 fusion classes in D_3.

Fusion Classes of groups of order 8

Fusion classes of D_4:

$$D_4 \cong < a, b; a^4 = b^2 = 1, bab = a^{-1} >$$

Fusion Classes are:

$$\overline{cl(1)} = \{1\}$$
$$\overline{cl(a)} = \{a^i; (i, 2) = 1, 1 \le i < 4\}$$
$$\overline{cl(a^2)} = \{a^2\}$$
$$\overline{cl(b)} = \{a^i b; 1 \le i \le 4\}.$$

Therefore, there are 4 fusion classes in D_4.

Fusion Classes of Q_3:

$$Q_3 \cong < a, b: a^4 = 1, b^2 = a^2, aba = b >$$

Thus, the fusion classes are:

$$\overline{cl(1)} = \{1\}$$
$$\overline{cl(a)} = \{a, b, a^3, b^3, ab, a^3b\}$$
$$\overline{cl(a^2)} = \{a^2\}$$

Hence the total number of fusion classes are 3.

Fusion Classes of groups of order 10

Fusion Classes of D_5:

$$D_5 \cong < a, b: a^5 = b^2 = 1, bab = a^{-1} >$$

Fusion Classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{a^i; (i, 5) = 1\},$$
$$\overline{cl(b)} = \{a^i b; 1 \le i \le 5\}.$$

So there are 3 fusion classes.

Fusion Classes of groups of order 12

Fusion Classes of $Z_3 \rtimes Z_4$:

$$Z_3 \rtimes Z_4 \cong < a, b : a^3 = b^4 = 1, b^{-1}ab = a^2 >$$

Fusion Classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{a^i ; 1 \leq i < 3\},$$
$$\overline{cl(b)} = \{a^i b^j ; 2 \nmid j, 0 \leq i \leq 2, 1 \leq j \leq 3\},$$
$$\overline{cl(b^2)} = \{b^2\},$$
$$\overline{cl(ab^2)} = \{a^i b^2 ; 1 \leq i \leq 2\}.$$

Hence there are 5 fusion classes.

Fusion Classes of A_4:

$$A_4 \cong < a, b, c : a^2 = b^2 = c^3 = 1, ba = ab, ca = abc, cb = ac >$$

Fusion Classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{a^i b^j ; 0 \leq i, j \leq 1, \quad i + j \neq 0\},$$
$$\overline{cl(c)} = \left\{a^i b^j c^k ; 0 \leq i, j \leq 1, \quad k \in \{1,2\}\right\}.$$

Thus, total number of fusion classes are 3.

Fusion Classes of D_6:

$$D_6 \cong < a, b : a^6 = b^2 = 1, bab = a^{-1} >$$

Fusion Classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \left\{a^i ; \quad i \in \{1,5\}\right\},$$
$$\overline{cl(a^2)} = \left\{a^i ; \quad i \in \{2,4\}\right\},$$
$$\overline{cl(a^3)} = \{a^3\},$$
$$\overline{cl(b)} = \{a^i b ; 0 \leq i \leq 5\}.$$

Total number of fusion classes are 5.

Fusion Classes of groups of order 14

Fusion Classes of D_7:

$$D_7 = < a, b : a^7 = b^2 = 1, bab = a^{-1} >$$

Fusion Classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{a^i ; 1 \leq i \leq 6\},$$
$$\overline{cl(b)} = \{a^i b ; 0 \leq i \leq 6\}.$$

So there are 3 fusion classes.

Fusion Classes of groups of order 16

Fusion classes of D_8:

Fusion Classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{a^i ; (i, 2) = 1, 1 \leq i < 8\},$$
$$\overline{cl(a^2)} = \{a^2, a^6\},$$
$$\overline{cl(a^4)} = \{a^4\},$$
$$\overline{cl(b)} = a^i b ; 0 \leq i \leq 7.$$

Total number of fusion classes are 5.

Fusion Classes of G:

$$G \cong < a, b: a^8 = b^2 = 1, bab = a^3 >$$

Fusion Classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{a^i; (2, i) = 1, 1 \leq i \leq 8\},$$
$$\overline{cl(a^2)} = \{a^{2i}; (2, i) = 1, 1 \leq i \leq 4\},$$
$$\overline{cl(a^4)} = \{a^4\},$$
$$\overline{cl(b)} = \left\{a^i b; \frac{2}{i}, 0 \leq i < 8\right\},$$
$$\overline{cl(ab)} = \{a^i b; (2, i) = 1, 1 \leq i \leq 8\}.$$

So there are 6 fusion classes.

Fusion Classes of Q_{16}:

$$Q_{16} \cong < a, b: a^8 = 1, a^4 = b^2, \quad aba = b >$$

Fusion Classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{a^i; (2, i) = 1, 1 \leq i \leq 8\},$$
$$\overline{cl(a^2)} = \{a^2, a^6\},$$
$$\overline{cl(a^4)} = \{a^4\},$$
$$\overline{cl(b)} = \{a^i b^j; 0 \leq i \leq 3, \quad j = 1, 3\}.$$

Therefore, there are 5 fusion classes.

Fusion classes of $D_4 \times Z_2$:

$$D_4 \times Z_2 \cong < a, b, c: a^4 = b^2 = c^2 = 1, ac = ca, bc = cb, bab = a^{-1} >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \left\{a^i c^j; \quad i \in \{1,3\}, \quad j \in \{0,1\}\right\},$$
$$\overline{cl(b)} = \left\{a^k b c^l; 0 \leq k \leq 3, \quad l \in \{0,1\}\right\},$$
$$\overline{cl(c)} = \{c, a^2 c\},$$
$$\overline{cl(a^2)} = \{a^2\}.$$

Therefore, there are 5 fusion classes.

Fusion classes of $Q_3 \times Z_2$:

$$Q_3 \times Z_2 \cong < a, b, c: a^4 = b^4 = c^2 = 1, b^2 = a^2, ba = a^3 b, ac = ca, bc = cb >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{a, a^3, b, b^3, ab, ac, a^3 b, a^3 c, bc, abc, a^2 bc, a^3 bc\},$$
$$\overline{cl(a^2)} = \{a^2\},$$
$$\overline{cl(c)} = \{c, a^2 c\}.$$

Total number of fusion classes are 4.

Fusion classes of $Modular - 16$:

$$Modular - 16 = G \cong < a, b: a^8 = b^2 = 1, ab = ba^5 >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \left\{a^i b^j; 2 \nmid i, 1 \leq i \leq 8, \quad j \in \{0,1\}\right\},$$
$$\overline{cl(b)} = \{b, a^4 b\},$$
$$\overline{cl(a^2)} = \{a^2, a^6\},$$
$$\overline{cl(a^2 b)} = \{a^2 b, a^6 b\},$$

$$\overline{cl(a^4)} = \{a^4\}.$$

Total number of fusion classes are 6.

Fusion classes of *B*:

$$B \cong < a,b: a^4 = b^4 = 1, ab = ba^3 >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \left\{a^i b^j; \quad i \in \{1,3\}, \quad j \in \{0,2\}\right\},$$
$$\overline{cl(b)} = \left\{a^l b^m; 0 \le l \le 3, \quad m \in \{1,3\}\right\},$$
$$\overline{cl(a^2)} = \{a^2\},$$
$$\overline{cl(b^2)} = \{b^2\},$$
$$\overline{cl(a^2b^2)} = \{a^2b^2\}.$$

Total number of fusion classes are 6.

Fusion classes of *K*:

$$K \cong < a,b,c: a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, cb = a^2bc >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \left\{a^i; \quad i \in \{1,3\}\right\},$$
$$\overline{cl(b)} = \{b, c, a^2b, a^2c, abc, a^3bc\},$$
$$\overline{cl(a^2)} = \{a^2\},$$
$$\overline{cl(ab)} = \{ab, ac, bc, a^3b, a^3c, a^2bc\}.$$

Total number of fusion classes are 5.

Fusion classes of $G_{4,4}$:

$$G_{4,4} \cong < a,b: a^4 = b^4 = (ab)^2 = 1, ab^3 = ba^3 >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{a, a^3, b, b^3, ab^2, a^2b, a^2b^3, a^3b^2\},$$
$$\overline{cl(a^2)} = \{a^2, b^2\},$$
$$\overline{cl(ab)} = \{ab, a^3b, ab^3, a^3b^3\},$$
$$\overline{cl(a^2b^2)} = \{a^2b^2\}.$$

Total number of fusion classes are 5.

Fusion Classes of groups of order 18

Fusion classes of D_9

$$D_9 \cong < a,b: a^9 = b^2 = 1, bab = a^{-1} >$$

Fusion Classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{a^i; 3 \nmid i, 1 \le i \le 8\},$$
$$\overline{cl(a^3)} = \left\{a^i; \quad i \in \{3,6\}\right\},$$
$$\overline{cl(b)} = \{a^i b; 0 \le i \le 8\}.$$

Total number of fusion classes are 4.

Fusion classes of $S_3 \times Z_3$:

$$S_3 \times Z_3 \cong < a,b,c: a^3 = b^2 = c^3 = 1, ba = a^{-1}b, ac = ca, bc = cb >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{a^j b; 0 \le j \le 2\},$$
$$\overline{cl(b)} = \{a^j b; 0 \le j \le 2\},$$

$$\overline{cl(c)} = \{c^k; 1 \le k \le 2\},$$
$$\overline{cl(ac)} = \{a^i c^j; 1 \le i, j \le 2\},$$
$$\overline{cl(bc)} = \{a^i b c^j; 0 \le i \le 2, 1 \le j \le 2\}.$$

Total number of fusion classes are 6.

Fusion classes of $(Z_3 \times Z_3) \rtimes Z_2$:

$$(Z_3 \times Z_3) \rtimes Z_2 \cong < a, b, c: a^2 = b^3 = c^3 = 1, bc = cb, bab = a, cac = a >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{ab^i c^j; 0 \le i, j \le 2\},$$
$$\overline{cl(b)} = \{b^k c^l; 0 \le k, l \le 2, \quad k + l \ne 0\}.$$

Total number of fusion classes are 3.

Fusion Classes of groups of order 20

Fusion classes of D_{10}:

$$D_{10} \cong < a, b: a^{10} = b^2 = 1, bab = a^{-1} >$$

Fusion Classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{a^i; 2 \nmid i, 5 \nmid i, 1 \le i < 10\},$$
$$\overline{cl(a^2)} = \{a^i; 2/i, 1 \le i < 10\},$$
$$\overline{cl(a^5)} = \{a^5\},$$
$$\overline{cl(b)} = \{a^i b; 0 \le i \le 9\}.$$

Total number of fusion classes are 5.

Fusion classes of Fr_{20}:

$$Fr_{20} \cong Z_5 \rtimes Z_4 \cong < a, b: a^4 = b^5 = 1, ba = ab^2 >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{ab^i; 0 \le i \le 4\},$$
$$\overline{cl(b)} = \{b^j; 1 \le j \le 4\},$$
$$\overline{cl(a^2)} = \{a^2 b^i; 0 \le i \le 4\},$$
$$\overline{cl(a^3)} = \{a^3 b^i; 0 \le i \le 4\}.$$

Total number of fusion classes are 5.

Fusion classes of $Z_5 \rtimes Z_4$:

$$Z_5 \rtimes Z_4 \cong < a, b: a^4 = b^5 = 1, bab = a >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{a^i b^j; \quad i \in \{1,3\}, 0 \le j \le 4\},$$
$$\overline{cl(b)} = \{b^j; 1 \le j \le 4\},$$
$$\overline{cl(a^2)} = \{a^2\},$$
$$\overline{cl(a^2 b)} = \{a^2 b^i; 1 \le i \le 4\}.$$

Total number of fusion classes are 5.

Fusion Classes of groups of order 21

Fusion classes of Fr_{21}:

$$Fr_{21} \cong Z_7 \rtimes Z_3 \cong < a, b: a^3 = b^7 = 1, ba = ab^2 >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{ab^i; 0 \le i \le 6\},$$

$$\overline{cl(b)} = \{b^j; 1 \le j \le 6\},$$
$$\overline{cl(a^2)} = \{a^2b^i; 0 \le i \le 6\}.$$

Total number of fusion classes are 4.

Fusion Classes of groups of order 22

Fusion Classes of D_{11}:

$$D_{11} =< a, b: a^{11} = b^2 = 1, bab = a^{-1} >$$

Fusion Classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{a^i; 1 \le i \le 10\},$$
$$\overline{cl(b)} = \{a^ib; 0 \le i \le 10\}.$$

So there are 3 fusion classes.

Fusion classes of groups of order 24

Fusion classes of $S_3 \times Z_4$:

$$S_3 \times Z_4 \cong< x, y, z: x^3 = y^2 = z^4 = 1, x^y = x^{-1}, xz = zx, yz = zy >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(x)} = \{x^i; 1 \le i \le 2\},$$
$$\overline{cl(y)} = \left\{x^jyz^k; 0 \le j \le 2, \quad k \in \{0,2\}\right\},$$
$$\overline{cl(z)} = \left\{z^u; u \in \{1,3\}\right\},$$
$$\overline{cl(xz)} = \left\{x^iz^j; 1 \le i \le 2, \quad j \in \{1,3\}\right\},$$
$$\overline{cl(yz)} = \left\{x^iyz^k; 0 \le i \le 2, \quad k \in \{1,3\}\right\},$$
$$\overline{cl(z^2)} = \{z^2\},$$
$$\overline{cl(xz^2)} = \{x^iz^2; 1 \le i \le 2\}.$$

Total number of fusion classes are 8.

Fusion classes of $S_3 \times Z_2 \times Z_2$:

$$S_3 \times Z_2 \times Z_2 \cong< x, y, z, w: x^3 = y^2 = z^2 = w^2 = 1, x^y = x^{-1}, xz = zx, xw = wx, yz = zy, yw = wy, zw = wz >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(x)} = \{x^i; 1 \le i \le 2\},$$
$$\overline{cl(y)} = \{x^jyz^kw^l; 0 \le j \le 2, 0 \le k, l \le 1\},$$
$$\overline{cl(z)} = \{z^uw^v; 0 \le u, v \le 1, \quad u + v \ne 0\},$$
$$\overline{cl(xz)} = \{x^iz^jw^k; 1 \le i \le 2, 0 \le j, k \le 1, \quad j + k \ne 0\}.$$

Total number of fusion classes are 5.

Fusion classes of $D_4 \times Z_3$:

$$D_4 \times Z_3 \cong< x, y, z: x^3 = y^4 = z^2 = 1, y^z = y^{-1}, xy = yx, xz = zx >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(x)} = \{x^i; 1 \le i \le 2\},$$
$$\overline{cl(y)} = \left\{y^j; \quad j \in \{1,3\}\right\},$$
$$\overline{cl(z)} = \{y^kz; 0 \le k \le 3\},$$
$$\overline{cl(y^2)} = \{y^2\},$$
$$\overline{cl(xy)} = \left\{x^iy^j; 1 \le i \le 2, \quad j \in \{1,3\}\right\},$$
$$\overline{cl(xz)} = \{x^iy^jz; 1 \le i \le 2, 0 \le j \le 3\},$$
$$\overline{cl(xy^2)} = \{x^iy^2; 1 \le i \le 2\}.$$

Total number of fusion classes are 8.

Fusion classes of $Q \times Z_3$:

$$Q \times Z_3 \cong < x, y, z: x^4 = z^3 = 1, x^2 = y^2, x^y = x^{-1}, yz = zy, xz = zx >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(x)} = \{x^i y^j; 0 \le i \le 3, 0 \le j \le 1\},$$
$$\overline{cl(z)} = \{z^i; 1 \le i \le 2\},$$
$$\overline{cl(xz)} = \{x^i y^j z^k; 0 \le i \le 3, 0 \le j \le 1, 1 \le k \le 2, (i,j) \ne (0,0), (2,0)\},$$
$$\overline{cl(x^2)} = \{x^2\},$$
$$\overline{cl(x^2 z)} = \{x^2 z^i; 1 \le i \le 1\}.$$

So total number of fusion classes are 6.

Fusion classes of $A_4 \times Z_2$:

$$A_4 \times Z_2 \cong < x, y, z, w: x^2 = y^2 = z^3 = w^2 = 1, xy = yx, zx = xyz, xw = wx, xz = zy, yw = wy, zw = wz >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(x)} = \{x^i y^j; 0 \le i, j \le 1, \quad i + j \ne 0\},$$
$$\overline{cl(z)} = \{x^l y^m z^n; 0 \le l, m \le 1, 1 \le n \le 2\},$$
$$\overline{cl(w)} = \{w\},$$
$$\overline{cl(xw)} = \{x^i y^j w; 0 \le i, j \le 1, \quad i + j \ne 0\},$$
$$\overline{cl(zw)} = \{x^l y^m z^n w; 0 \le l, m \le 1, 1 \le n \le 2\}.$$

So total number of fusion classes are 6.

Fusion classes of Q_{12}:

$$Q_{12} = < a, b: a^{12} = 1, a^6 = b^2, b^{-1}ab = a^{-1} >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{a^i; 1 \le i \le 12, (i, 12) = 1\},$$
$$\overline{cl(a^2)} = \left\{a^i; \quad i \in \{2,10\}\right\},$$
$$\overline{cl(a^3)} = \left\{a^i; \quad i \in \{3,9\}\right\},$$
$$\overline{cl(a^4)} = \left\{a^i; \quad i \in \{4,8\}\right\},$$
$$\overline{cl(a^6)} = \{a^6\},$$
$$\overline{cl(b)} = \{a^j b; 0 \le j \le 11\}.$$

So total number of fusion classes are 7.

Fusion classes of D_{12}:

$$D_{12} = < a, b: a^{12} = b^2 = 1, bab = a^{-1} >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(a)} = \{a^i; 1 \le i \le 12, (i, 12) = 1\},$$
$$\overline{cl(a^2)} = \left\{a^i; \quad i \in \{2,10\}\right\},$$
$$\overline{cl(a^3)} = \left\{a^i; \quad i \in \{3,9\}\right\},$$
$$\overline{cl(a^4)} = \left\{a^i; \quad i \in \{4,8\}\right\},$$
$$\overline{cl(a^6)} = \{a^6\},$$
$$\overline{cl(b)} = \{a^j b; 0 \le j \le 11\}.$$

So total number of fusion classes are 7.

Fusion classes of $Z_6 \rtimes Z_4$:

$$Z_6 \rtimes Z_4 \cong < x, y: x^4 = y^6 = 1, yxy = x >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(x)} = \{x^i y^j; \quad i \in \{1,3\}, 0 \le j \le 5\},$$
$$\overline{cl(y)} = \left\{x^k y^l; \quad k \in \{0,2\}, \quad l \in \{1,5\}\right\},$$
$$\overline{cl(x^2)} = \{x^2\},$$
$$\overline{cl(y^2)} = \left\{y^j; \quad j \in \{2,4\}\right\},$$
$$\overline{cl(y^3)} = \left\{x^i y^3; \quad i \in \{0,2\}\right\},$$
$$\overline{cl(x^2y^2)} = \left\{x^2 y^j; \quad j \in \{2,4\}\right\}.$$

So total number of fusion classes are 7.

Fusion classes of $Z_3 \rtimes Z_8$:

$$Z_3 \rtimes Z_8 \cong < x, y: x^3 = y^8 = 1, yxy^{-1} = x^{-1} >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(x)} = \{x^i; 1 \le i \le 2\},$$
$$\overline{cl(y)} = \{x^k y^l; 0 \le k \le 2, (2, l) = 1, 1 \le l \le 8\},$$
$$\overline{cl(y^2)} = \left\{y^j; \quad j \in \{2,6\}\right\},$$
$$\overline{cl(y^4)} = \{y^4\},$$
$$\overline{cl(xy^2)} = \left\{x^i y^j; 1 \le i \le 2, j \in \{2,6\}\right\},$$
$$\overline{cl(xy^4)} = \{x^i y^4; 1 \le i \le 2\}.$$

So total number of fusion classes are 7.

Fusion classes of$Z_3 \rtimes Q$:

$$Z_3 \rtimes Q \cong < x, y, z: x^2 = y^6 = z^2 = 1, xy = yx, xz = zx, (zy)^2 = 1 >$$

Fusion classes are:

$$\overline{cl(1)} = \{1\},$$
$$\overline{cl(x)} = \{x\},$$
$$\overline{cl(y)} = \left\{x^l y^m; 0 \le l \le 1, \quad m \in \{1,5\}\right\},$$
$$\overline{cl(z)} = \{x^i y^j z; 0 \le i \le 1, 0 \le j \le 5\},$$
$$\overline{cl(y^2)} = \{y^j; \quad j \in \{2,4\}\},$$
$$\overline{cl(y^3)} = \{x^i y^3; 0 \le i \le 1\},$$
$$\overline{cl(xy^2)} = \left\{xy^j; \quad j \in \{2,4\}\right\}.$$

Thus, there are 7 fusion classes.

CONCLUSION

Sr. No.	Group	order	$\lvert Aut(G)\rvert$	Number of fusion classes
				$k(G)$
	D_3	6	6	3
	D_4	8	8	4
	Q_3	8	24	3
	D_5	10	20	3
	$Z_3 \rtimes Z_4$	12	12	5
	A_4	12	24	3
	D_6	12	12	5
	D_7	14	42	3
	D_8	16	32	5
	G	16	16	6
	Q_{16}	16	32	5
	$D_4 \times Z_2$	16	64	5

	$Q_3 \times Z_2$	16	96	4
	$Modular-16$	16	16	6
	B	16	32	6
	K	16	48	5
	$G_{4,4}$	16	32	5
	D_9	18	54	4
	$S_3 \times Z_3$	18	12	6
	$(Z_3 \times Z_3) \rtimes Z_2$	18	432	3
	D_{10}	20	40	5
	Fr_{20}	20	20	5
	$Z_5 \rtimes Z_4$	20	40	5
	Fr_{21}	21	42	4
	D_{11}	22	110	3
	$S_3 \times Z_4$	24	24	8
	$S_3 \times Z_2 \times Z_2$	24	144	5
	$D_4 \times Z_3$	24	48	6
	$Q \times Z_3$	24	48	6
	$A_4 \times Z_2$	24	24	6
	Q_{12}	24		
	D_{12}	24	48	7
	$Z_6 \rtimes Z_4$	24	48	7
	$Z_3 \rtimes Z_8$	24	24	7
	$Z_3 \rtimes Q$	24	48	7

Theorem *The bounds of fusion classes of metabelian groups of order less than equal to 24 and given by* $3 \leq k(G) \leq 8$.

Proof. Obvious from the observation obtained earlier.

REFERENCES

[1]. Muniya, Harsha Arora, Mahender Singh. Automorphisms of some $p-$ groups of order p^4. International Journal of Innovative Science and Research Technology, 7(7), 56—62(2022).

[2]. Muniya, Harsha Arora, Mahender Singh. Number of Automorphisms of some Non-Abelian $p-$ groups of order p^4. Applications and Applied Mathematics(AAM), 17(3), 83—96(2022)

[3]. Rahman, S. F. A., & Sarmin, N. H. (2012). Metabelian groups of order at most 24. *Menemui Matematik (Discovering Mathematics)*, *34*(1), 77-94.

[4]. Sarmin, N. H., Gambo, I., and Omer, S. M. S. (2015). The conjugacy classes of metabelian groups of order at most 24. *Jurnal Teknologi*, *77*(1):139-143