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Solution of Fractional Differential Equation with Certain Boundary Conditions

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ABSTRACT

In this paper we investigate the existence and uniqueness of solution of fractional differential equation with boundary conditions by using Banach, Schaefer's and Leray-Schauder nonlinear alternative fixed point theorems respectively, also these investigation leads us to extend the work of Ravip. AGARWAL 1.

Keywords: Fractional differential equations, Existence and uniqueness solutions, Boundary conditions.

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1. INTRODUCTION:

This paper studies the existence and uniqueness of solutions of fractional differential equation with boundary conditions,

$$^{c}D^{\alpha}y(t) = f(t,y), \ for \ each \ \ t \in J = [0,T], \ n-1 < \alpha \le n, n \in N, n > 2 \qquad (1.1)$$

$$y(0) = y_0, y'(0) = y_1, y''(0) = y_2, ... \ , y^{(n-2)}(0) = y_{n-2}, y^{(n-1)}(T) = y_{n-1} \qquad (1.2)$$

Where $f : JxR \to R$ is a continuous function and $y_0, y_1, y_2, ..., \ y_{n-2}, y_{n-1}$ are real constants, $^{c}D^{\alpha}$ is the Caputo fractional derivative. RAVIP. AGARWAL et al [1] Studied the existence and uniqueness of solution for boundary value problems, for fractional differential equations.

$$^{c}D^{\alpha}y(t) = f(t,y), \ for \ \ each \ \ t \in J = [0,T], \ 2 < \alpha \le 3$$

$$y(0) = y_0, y'(0) = y_0^{*}, y''(T) = y_T \qquad (1.3)$$

Fractional differential equation can be extensively applied to various disciplines such as Physics, mechanics, and engineering see [8,10]. Indeed, we can find numerous applications in viscoelasticity, electro chemistry, contro, porous media, electromagnetic etc. see [7,9,11,12,13]. Hence, in recent years fractional differential equations have been of great interest, and there have been many results on existence and uniqueness of the solution of boundary value problems for fractional differential equations, see [14,15]. There has been a significant progress in the investigation of fractional differential and partial differential equations in recent years; see the monographs of kilbas et al [2], [4].
Our work is to extend the existence and uniqueness solution of the problem where given by [1].

In this paper, we give some theorems, firstly Banach fixed point theorem, secondly Schaefer's fixed point theorem, and thirdly Leray – Schauder nonlinear alternative. Finally, we present an example illustrating the applicability of the imposed conditions.

2- PRELIMINARIES

In this section, we introduce the notation, definitions, and preliminary facts which are used in this paper. By C(J,R) we denote the Banach space of all continuous functions from J into R with the norm

$\| y \|_{\infty} := \sup\{| \quad y, t) |: T \in J\}.$

Definition 2.1. [2,3].

The fractional (arbitrary) order integral of the function

$h \in L'([a,b], R)$ *of order* $\alpha \in R_{+}$ *is defined by*

$$I_a^{\alpha} h(t) = \int_{\alpha}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function. When a=0, we write

$$I^{\alpha} h(t) = [h * \vartheta_{\alpha}](t), \text{ where } \quad \vartheta_{\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \quad \text{for } \ t > 0,$$

$$\text{and } \ \vartheta_{\alpha}(t) = 0 \ \text{ for } \ t \le 0, \text{ and } \ \vartheta_{\alpha} \to \delta(t) \ \text{ as } \ \alpha \to 0, \text{Where } \ \delta \text{ is the delta function }.$$

Definition 2.2. [2,3].

For a function h defined on the interval [a,b], the αth Riemann-Liouville fractional-order derivative of h, is defined by

$$(D_{a+}^{\alpha} h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_{a}^{t} (t-s)^{n-\alpha-1} h(s)ds.$$

Here n=[α]+1 and [α] denotes the integer part of α .

Definition 2.3. [4].

For a function h defined on the interval [a,b], the Caputo fractional-order derivative of order α of h, is defined by

$$({}^{c}D_{a+}^{\alpha} h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-s)^{n-\alpha-1} h^{(n)}(s)ds$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer point of α .

3. EXISTENCE OF SOLUTIONS

In the section, we give some definitions, lemmas which is useful in our work.

Definition 3.1:

A function $y \in C^{n-1}(J, R)$, with its α - derivative existing on J is said to be a solution of (1.1) – (1.2) if y satisfies the equation ${}^{c}D^{\alpha} y(t) = f(t, y(t))$ on J and the conditions

$y(0) = y_0, y'(0) = y_1, y''(0) = y_2, ... , y^{(n-2)}(0) = y_{n-2}, y^{(n-1)}(T) = y_{n-1}.$

For the existence of the solutions for the problem (1.1) – (1.2), we need the following lemmas:

Lemma 3.1. [5]:

Let α >0, then the differential equation ${}^{c}D^{\alpha} h(t) = 0$ has a solution $h(t) = c_0 + c_1 t + c_2 t^2 + ... + c_n t^n, c_i \in R, \ i = 0,1,2,....., \ n, n = [\alpha] + 1$

Lemma 3.2 [5]:

Let α >0, then

$I^{\alpha}\, {}^{c}D^{\alpha} h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + ... + c_n t^n$, *for some* $c_i \in R, \ i = 0,1,2,....., \ n, n = [\alpha] + 1$ As a consequence of lemmas 3.1 and 3.2 we have the following result.

Lemma 3.3:

Let $n - 1 < \alpha \le n, n > 2, n \in N$ *and let* $h: J \to R$ be continuous. A function y is a solution of the fractional integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds - \frac{t^{n-1}}{(n-1)!\,\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n}h(s)ds +$$

$$y_0 + y_1 t + \frac{y_2 t^2}{2!} + \dots + \frac{y_{n-2}t^{n-2}}{(n-2)!} + \frac{y_{n-1}t^{n-1}}{(n-1)!} \qquad (3.1)$$

if and only if y is a solution of the fractional BVP

$$^cD^\alpha y(t) = h(t), t \in J \qquad (3.2)$$

$$y(0) = y_0, y'(0) = y_1, y''(0) = y_2, \dots\dots\dots, y^{(n-2)}(0) = y_{n-2}, y^{(n-1)}(T) = y_{n-1}. \qquad (3.3)`$$

Proof:
Assume y satisfies (3.2), then by lemma 3.2 we have

$$y(t) = c_0 + c_1 t + c_2 t^2 + \dots\dots\dots + c_{n-1}t^{n-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds.$$

By (3.3), all after a simple calculation we obtain

$$c_0 = y_0, c_1 = y_1, c_2 = \frac{y_2}{2!}, c_3 = \frac{y_3}{3!}, \dots\dots\dots, c_{n-2} = \frac{y_{n-2}}{(n-2)!}$$

and

$$y^{(n-1)}(T) = (n-1)!\,C_{n-1} + \frac{1}{\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n}h(s)ds = y_{n-1}$$

$$C_{n-1} = \frac{y_{n-1}}{(n-1)!} - \frac{\int (T-s)^{\alpha-h}h(s)ds}{(n-1)!\,\Gamma(\alpha-n+1)}$$

Hence we get equation (3.1). Conversely, it is clear that if y satisfies equation (3.1), then equations (3.2) – (3.3) hold. Our first result is based on the Banach fixed point theorem.

Theorem 3.1
Assume that

(I1) There exists a constant k>0 and $f : J \times R \to R$ continuous function such that

$$|f(t,z) - f(t,\bar{z})| \le k|z - \bar{z}|, \; for \;\; each \;\; t \in J, and \;\; all \;\; z, \bar{z} \in R$$

if

$$kT^\alpha\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{(n-1)!\,\Gamma(\alpha-n+2)}\right] < 1 \qquad (3.4)$$

Then the BVP (1.1) – (1.2) has a unique solution on J.

Proof:
Consider the operator

$$F : C(J,R) \to C(J,R)$$

defined by

$$F(y)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))\,ds - \frac{t^{n-1}}{(n-1)!\,\Gamma(\alpha-n+1)}\int_0^T (T-S)^{\alpha-n} f(s,y(s))\,ds$$

$$+ y_0 + y_1 t + \frac{y_2 t^2}{2!} + \dots\dots + \frac{y_{n-1}}{(n-1)!}t^{n-1} \qquad (*)$$

Clearly, the fixed points of the operator F are solutions of the problem (1.1) –(1.2). We shall use the Banach contraction principle to prove that F is a contraction.

Let x,y $\in$ C(J, R). Then for each t $\in$ J we have

$$|F(x)(t)-F(y)(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s))-f(s,y(s))|ds$$

$$+\frac{T^{n-1}}{(n-1)!\,\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n}|f(s,x(s))-f(s,y(s))|ds$$

$$\leq \frac{k\|x-y\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds+\frac{kT^{n-1}\|x-y\|_\infty}{(n-1)!\,\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n}ds$$

$$\leq kT^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(n-1)!\,\Gamma(\alpha-n+2)}\right]\|x-y\|_\infty$$

Thus

$$\|F(x)-F(y)\|_\infty \leq kT^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(n-1)!\,\Gamma(\alpha-n+2)}\right]\|x-y\|_\infty$$

Consequently, by (3.4) F is a contraction mapping. As consequence of the Banach fixed Point theorem, we deduce that F has a fixed point which is a solution of the problem (1.1) – (1.2).
The second result is based on Schaefer's fixed point theorem.

Theorem 3.2:
Assume that
(I2) The function $f: J \times R \to R$ is continuous.
(I3) There exists a constant M > 0 such that
$|f(t,z)| \leq M$ *for each* $t \in J$ *and all* $z \in R$.
Then the BVP (1.1) – (1.2) has at least one solution on J.
Proof:
We shall use Schaefer's fixed point theorem to prove that F defined by (*) has a fixed point. The proof will be given in several steps.
Step1: F is continuous
Let $\{y_n\}$ be a sequence such that $y_n \to y$ in C(J,R). Then for each t $\in$ J.

$$|F(y_n)(t)-F(y)(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y_n(s))-f(s,y(s))|ds$$

$$+\frac{T^{n-1}}{(n-1)!\,\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n}|f(s,y_n(s))-f(s,y(s))|ds$$

Since f is continuous, we have

$$\|F(y_n)-F(y)\|_\infty \to 0 \ \text{as} \ n\to\infty.$$

Step2: F maps the bounded sets into bounded sets in C(J,R).
Indeed, it is enough to show that for any η>0 there exists a positive constant ℓ such that for each $y \in B_\eta = \{y \in C(J,R): \|y\|_\infty \leq \eta\}$ *we have* $\|F(y)\|_\infty \leq \ell$. By (I3) we have for each t $\in$ J

$$|F(y)(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y(s))|ds+\frac{T^{n-1}}{(n-1)!\,\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n}|f(s,y(s))|dy$$

$$+|y_0|+|y_1|T+\frac{|y_2|}{2!}T^2+\dots+\frac{|y_{n-2}|}{(n-2)}T^{n-2}+\frac{|y_{n-1}|}{(n-1)!}T^{n-1}$$

$$\leq \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds+\frac{MT^{n-1}}{(n-1)!\,\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n}ds$$

$$+|y_0|+|y_1|T+\frac{|y_2|}{2!}T^2+\dots+\frac{|y_{n-1}|}{(n-1)!}T^{n-1}$$

$$\leq \frac{MT^{\alpha}}{\Gamma(\alpha+1)}+\frac{MT^{\alpha}}{(n-1)!\,\Gamma(\alpha-n+2)}+|y_0|+|y_1|T+\frac{|y_2|}{2!}T^2+\dots+\frac{|y_{n-1}|}{(n-1)!}T^{n-1}$$

Thus

$$\|F(y)\|_\infty \leq \frac{MT^{\alpha}}{\Gamma(\alpha+1)}+\frac{MT^{\alpha}}{(n-1)!\,\Gamma(\alpha-n+2)}+|y_0|+|y_1|T+\frac{|y_2|}{2!}T^2+\dots+\frac{|y_{n-1}|}{(n-1)!}T^{n-1}=\ell$$

Step 3: F maps the bounded sets into the equicontinuos sets of C(J,R).

Let $t_1, t_2 \in J, t_1 < t_2, B_\eta$ be bounded set of C(J,R) as in step2, and let $y \in B_\eta$. Then

$$| F(y)(t_2) - F(y)(t_1) | \leq | \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] f(s, y(s))\, ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, y(s))\, ds - \frac{t_2^{n-1} - t_1^{n-1}}{(n-1)!\,\Gamma(\alpha - n + 1)} \int_0^T (T - s)^{\alpha-n} f(s, y(s))\, ds \,|$$

$$+ | y_0 | + | y_1 | (t_2 - t_1) + \frac{|y_2|}{2!}(t_2^2 - t_1^2) + ... + \frac{| y_{n-1} |}{(n-1)!}(t_2^{n-1} - t_1^{n-1})$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] | f(s, y(s)) |\, ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \left| f(s, y(s)) \right| ds + \frac{t_2^{n-1} - t_1^{n-1}}{(n-1)!\,\Gamma(\alpha - n + 1)} \int_0^T (T - s)^{\alpha-n} | f(s, y(s)) |\, ds +$$

$$| y_0 | + | y_1 | (t_2 - t_1) | + \frac{|y_2|}{2!}(t_2^2 - t_1^2) + ... + \frac{| y_{n-1} |}{(n-1)!}(t_2^n - t_1^n)$$

$$\leq \frac{M}{\Gamma(\alpha + 1)} \left[-(t_2 - t_1)^\alpha + t_2^\alpha - t_1^\alpha \right] + \frac{M}{\Gamma(\alpha + 1)} (t_2 - t_1)^\alpha$$

$$+ \frac{M\left(t_2^{n-1} - t_1^{n-1}\right)}{(n-1)!\,\Gamma(\alpha - n + 2)} T^{\alpha-n+1} + | y_1 | (t_2 - t_1) + \frac{|y_2|}{2!}(t_2^2 - t_1^2) + ... + \frac{| y_{n-1} |}{(n-1)!}(t_2^{n-1} - t_1^{n-1})$$

$$= \frac{M}{\Gamma(\alpha + 1)} \left(t_2^\alpha - t_1^\alpha\right) + \frac{M\left(t_2^{n-1} - t_1^{n-1}\right)}{(n-1)!\,\Gamma(\alpha - n + 2)} T^{\alpha-n+1} + | y_1 | (t_2 - t_1) + \frac{|y_2|}{2!}(t_2^2 - t_1^2) + ... + \frac{| y_{n-1} |}{(n-1)!}(t_2^{n-1} - t_1^{n-1}).$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $F : C(J, R) \to C(J, R)$ is completely continuous.

Step4: A priori bounds

Now it remains to show that the set

$E = \{ y \in C(J, R) : y = \lambda F(y) \text{ for some } 0 < \lambda < 1 \}$ *is bounded*

Let $y \in E$, *then* $y = \lambda F(y)$ *for some* $0 < \lambda < 1$. *Thus for each* $t \in J$ *we have*

$$y(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, y(s))\, ds - \frac{\lambda t^{n-1}}{(n-1)!\,\Gamma(\alpha - n + 1)} \int_0^T (T - s)^{\alpha-n} f(s, y(s))\, ds +$$

$$\lambda y_0 + \lambda y_1 t + \frac{\lambda y_2 t^2}{2!} + ... + \frac{\lambda y_{n-2} t^{n-2}}{(n-2)!} + \lambda \frac{y_{n-1}}{(n-1)!} t^{n-1}$$

This implies by $(I3)$ *that for each* $t \in J$ *we have*

$$| y(t) | \leq \frac{M}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} ds + \frac{MT^{n-1}}{(n-1)!\,\Gamma(\alpha - n + 1)} \int_0^T (T - s)^{\alpha-n} ds +$$

$$| y_0 | + | y_1 | T + \frac{|y_2|}{2} T^2 + ... + \frac{| y_{n-1} |}{(n-1)!} T^{n-1}$$

Thus for every $t \in J$ *we have*

$$\| y \|_\infty \leq \frac{MT^\alpha}{\Gamma(\alpha + 1)} + \frac{MT^\alpha}{(n-1)!\,\Gamma(\alpha - n + 2)} + | y_0 | + | y_1 | T + \frac{|y_2|}{2!} T^2 + ... + \frac{| y_{n-1} |}{(n-1)!} T^{n-1} := S$$

This shows that the set E is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1.1) – (1.2).

In the following theorem we shall give an existence result for the problem (1.1) – (1.2) by means of an application of a Leray-Schauder type nonlinear alternative, where the condition (I3) is weakened.

Theorem 3.3:

Assume that (I2) and the following conditions hold

(I4) There exist $\phi_f \in L^1(J, R^+)$ and continuous and nondecreasing

$\Psi : [0,\infty) \to (0,\infty)$ *such that*

$|f(t,z)| \le \phi_f(t)\Psi(|z|)$ *for each* $t \in J$ *and all* $z \in R$.

(I5) There exists a number M > 0 such that

$$\frac{M}{\|I^\alpha \phi_f\|_{L^1}\Psi(M) + \frac{T^{n-1}}{(n-1)!}(I^{\alpha-n+1}\phi_f)(T)\Psi(M) + |y_0| + |y_1|T + \dots + \frac{|y_{n-1}|T^{n-1}}{(n-1)!}} > 1 \qquad (3.5)$$

Then BVP (1.1) – (1.2) has at least one solution on J

Proof: Consider the operator F defined in theorem 3.1 and 3.2. It can easily shown that F is continuous and completely continuous. For $\lambda \in [0,1]$ let y be such that for each $t \in J$ we have $y(t) = \lambda(Fy)(t)$. Then from (I4) – (I5) we have for each $t \in J$

$$|y(t)| \le \frac{1}{\Gamma(\alpha)}\left|\int_0^t (t-s)^{\alpha-1}\phi_f(s)\Psi(|y(s)|)ds + \frac{T^{n-1}}{(n-1)!\,\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n}\phi_f(s)\Psi(|y(s)|)ds + \right.$$

$$|y_0| + |y_1|T + \frac{|y_2|}{2}T^2 + \dots + \frac{|y_{n-1}|}{(n-1)!}T^{n-1}$$

$$\le \Psi(\|y\|_\infty)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_f(s)ds +$$

$$\Psi(\|y\|_\infty)\frac{T^{n-1}}{(n-1)!\,\Gamma(\alpha-n+1)}\int_0^T (T-s)^{\alpha-n}\phi_f(s)ds + |y_0| + |y_1|T + \frac{|y_2|}{2}T^2 + \dots + \frac{|y_{n-1}|}{(n-1)!}T^{n-1}$$

Thus

$$\frac{\|y\|_\infty}{\Psi(\|y\|_\infty)\|I^\alpha\phi_f\|_{L^1} + \frac{T^{n-1}}{(n-1)!}\left(I^{\alpha-n+1}\phi_f)(T)\right)\Psi(\|y\|_\infty) + |y_0| + |y_1|T + \frac{|y_2|}{2}T^2 + \dots + \frac{|y_{n-1}|}{(n-1)!}T^{n-1}} \le 1$$

Then by condition (I5), there exists M such that $\|y\|_\infty \ne M$.

Let $V = \{y \in C(J,R) : \|y\|_\infty < M\}$

The operator $F : \bar{V} \to C(J,R)$ is continuous and completely continuous. By the choice of $\bar{V}$, there exists no $y \in \partial V$ such that $y = \lambda F(y)$ for some $\lambda \in (0,1)$.

As a consequence of the nonlinear alternative of Leray-Schauder type [6], we deduce that F has a fixed point y in $\bar{V}$, which is a solution of the problem (1.1)-(1.2).
This completes the proof.

4. EXAMPLE

In this article we give an example [1] to illustrate the usefulness of our main results. Let us consider the following BVP

$$ {}^{c}D^{\alpha} y(t) = \frac{e^{-t} \, |y(t)|}{(9+e^{t})(1+|y(t)|)}, t \in [0,1], \, n-1 < \alpha \le n, n > 2, n \in N \qquad (4.1) $$

$$ y(0) = y_0, y'(0) = y_1, y''(0) = y_2, \ldots y^{(n-2)}(0) = y_{n-2}, y^{(n-1)}(T) = y_{n-1} \qquad (4.2) $$

$$ set \;\; f(t,x) = \frac{e^{-t} x}{(9+e^{t})(1+x)}, \quad (t,x) \in [0,\infty) \times J $$

Let $x, y \in [0,\infty)$ *and* $t \in J$. *Then we have*

$$ |f(t,x) - f(t,y)| = \frac{e^{-t}}{(9+e^{t})} \left| \frac{x}{1+x} - \frac{y}{1+y} \right| = \frac{e^{-t} \, |x-y|}{(9+e^{t})(1+x)(1+y)} $$

$$ \le \frac{e^{-t}}{(9+e^{t})} |x-y| \le \frac{1}{10} |x-y| $$

Hence the condition (I1) holds with k= $\frac{1}{10}$. We shall check that condition (3.4) is satisfied with T=1. Indeed

$$ kT^{\alpha} \left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{(n-1)!\,\Gamma(\alpha-n+2)} \right] < 1 \Leftrightarrow \frac{1}{\Gamma(\alpha+1)} + \frac{1}{(n-1)!\,\Gamma(\alpha-n+2)} < 10 \qquad (4.3) $$

We have

$$ \frac{1}{n!} \le \frac{1}{\Gamma(\alpha+1)} < \frac{1}{(n-1)!} \qquad (4.4) $$

$$ and \;\; \frac{1}{(n-1)!} \le \frac{1}{(n-1)!\,\Gamma(\alpha-n+2)} < C \qquad (4.5) $$

for an appropriately chosen constant C that will be specified (4.3) – (4.5) imply that

$$ \frac{1}{\Gamma(\alpha+1)} + \frac{1}{(n-1)!\,\Gamma(\alpha-n+2)} < \frac{1}{(n-1)!} + C < 10 \qquad (4.6) $$

Thus from (4.6) the positive constant C must satisfy

$$ C < \frac{10\,(n-1)! - 1}{(n-1)!} $$

From (4.5) we get

$$ \Gamma(\alpha - n + 2) > \frac{1}{10\,(n-1)! - 1} \qquad (4.7) $$

Which is satisfied for some $\alpha \in (n-1, n]$.

Then by theorem 3.1 the problem (4.1) – (4.2) has a unique solution on [0,1] for the values of α satisfying (4.7).

Remark: if $\alpha = 1$, we get the result of [16]

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